On the trivectors of a 6-dimensional symplectic vector space. III

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Abstract

Let V be a 6-dimensional vector space over a field $\mathbb{F}$, let f be a nondegenerate alternating bilinear form on V and let $Sp(V,f) \cong Sp_6(\mathbb{F})$ denote the symplectic group associated with (V,f). The group $GL(V)$ has a natural action on the third exterior power $\bigwedge^3 V$ of V and this action defines five families of nonzero trivectors of V. In this paper, we divide one of these five families into suborbits for the action of $Sp(V,f) \subseteq GL(V)$ on $\bigwedge^3 V$.

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1 Introduction

Throughout this paper, we assume that V is a 6-dimensional vector space over a field $\mathbb{F}$ and that f is a nondegenerate alternating bilinear form on V. We denote the symplectic group associated with (V,f) by $Sp(V,f)$. An ordered basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of V is called a *hyperbolic basis* of (V,f) if $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i,j \in \{1,2,3\}$. The elements of $Sp(V,f)$ are precisely those elements of $GL(V)$ that map hyperbolic bases to hyperbolic bases.

The *trivectors* of V are the elements of the third exterior power $\bigwedge^3 V$ of V. For every $\theta \in GL(V)$, there exists a unique $\bigwedge^3(\theta) \in GL(\bigwedge^3 V)$ such that $\bigwedge^3(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \theta(\bar{v}_3)$ for all $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$. Two trivectors α_1 and α_2 of V are called $GL(V)$-*equivalent* [resp., $Sp(V,f)$-*equivalent*] if there exists a $\theta \in GL(V)$ [resp., $\theta \in Sp(V,f)$] such that $\bigwedge^3(\theta)(\alpha_1) = \alpha_2$.

Let $\overline{\mathbb{F}}$ be a fixed algebraic closure of $\mathbb{F}$. For every quadratic extension $\mathbb{K}$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, let $\chi^*_{\mathbb{K}}$ be a fixed trivector of the form $\mu_1 \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \mu_2 \cdot \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_4) \wedge (\bar{v}_2 + \bar{v}_5) \wedge (\bar{v}_3 + \bar{v}_6)$, where $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ is some basis of V and $\mu_1, \mu_2 \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$ are chosen in such a way that $\mathbb{K} \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ defined by the irreducible quadratic polynomial $\mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1 = \mu_2(X^2 - (\mu_1 + 1 + \frac{\mu_1}{\mu_2})X + \frac{\mu_1}{\mu_2})$ of $\mathbb{F}[X]$. Obviously, different choices for μ_1, μ_2 and $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6)$ can give rise to distinct trivectors.

However, as shown in De Bruyn [3], the $GL(V)$-equivalence class of the trivector $\chi^*_{\mathbb{K}}$ only depends on $\mathbb{K}$ and not on the particular choices of μ_1, μ_2 and $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6)$.

In [8], Reichel obtained a complete classification of all $GL(V)$-equivalence classes of trivectors of V, assuming the base field $\mathbb{F}$ is the field of complex numbers. For perfect fields of cohomological dimension at most 1, the classification of the $GL(V)$-equivalence classes of trivectors of V can be found in Cohen and Helminck [1]. Revoy [9] obtained a complete classification of all $GL(V)$-equivalence classes of trivectors of V without imposing any restrictions on the underlying field $\mathbb{F}$. Their classification results can be found in the following proposition.

Proposition 1.1 ([1, 8, 9]) *Let $\{\bar{v}^*_1, \bar{v}^*_2, \ldots, \bar{v}^*_6\}$ be a fixed basis of V. Then every nonzero trivector of V is $GL(V)$-equivalent with precisely one of the following trivectors:*
(A) $\bar{v}^*_1 \wedge \bar{v}^*_2 \wedge \bar{v}^*_3$;
(B) $\bar{v}^*_1 \wedge \bar{v}^*_2 \wedge \bar{v}^*_3 + \bar{v}^*_1 \wedge \bar{v}^*_4 \wedge \bar{v}^*_5$;
(C) $\bar{v}^*_1 \wedge \bar{v}^*_2 \wedge \bar{v}^*_3 + \bar{v}^*_4 \wedge \bar{v}^*_5 \wedge \bar{v}^*_6$;
(D) $\bar{v}^*_1 \wedge \bar{v}^*_2 \wedge \bar{v}^*_4 + \bar{v}^*_2 \wedge \bar{v}^*_3 \wedge \bar{v}^*_5 + \bar{v}^*_3 \wedge \bar{v}^*_1 \wedge \bar{v}^*_6$;
(E) $\chi^*_{\mathbb{K}}$ *for some quadratic extension* $\mathbb{K}$ *of* $\mathbb{F}$ *contained in* $\overline{\mathbb{F}}$.

A nonzero trivector α of V is said to be of *Type (X)*, where $X \in \{A, B, C, D, E\}$, if it is $GL(V)$-equivalent with a trivector described in (X) of Proposition 1.1. The description of the trivectors of Type (E) in terms of the parameters μ_1 and μ_2 is taken from De Bruyn [3].

Popov [7, Section 3] obtained a complete classification of all $Sp(V, f)$-equivalence classes of trivectors of V, assuming the underlying field $\mathbb{F}$ is algebraically closed and of characteristic distinct from 2. Popov's method heavily relies on the decomposition of $\bigwedge^3 V$ as a direct sum of two subspaces W_1 and W_2 of respective dimensions 14 and 6, which is only valid if $char(\mathbb{F}) \neq 2$, and invokes a result of Igusa [6] regarding the $Sp(V, f)$-equivalence classes of trivectors contained in W_1. This classification result of Igusa assumes that the underlying field is algebraically closed and of characteristic distinct from 2. In view of their applications to hyperplanes and projective embeddings of symplectic dual polar spaces, the authors are interested in the classification of all $Sp(V, f)$-equivalence classes of trivectors, regardless of the structure of the underlying field.

The $Sp(V, f)$-equivalence classes of the trivectors of Type (A), (B) and (C) were determined in De Bruyn and Kwiatkowski [4] for any field $\mathbb{F}$. Those of Type (E) were classified in De Bruyn and Kwiatkowski [5] under the assumption that the corresponding quadratic field extension $\mathbb{K}/\mathbb{F}$ is separable. In the present paper, we give a classification of the $Sp(V, f)$-equivalence classes of trivectors of Type (D) without imposing any restrictions on the underlying field.

For every ordered basis $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ of V, we define

$$\gamma_B := \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2,$$

and call $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ the *base 3-space* of the trivector γ_B. The trivectors of Type (D) are precisely the trivectors of the form γ_B for some ordered basis B of V. By Lemma 3.1 of Section 3, the base 3-space $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is uniquely determined by γ_B. So, we can divide the trivectors of Type (D) into two disjoint classes, those whose base 3-space is totally isotropic and those whose base 3-space is not.

Now, let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a fixed hyperbolic basis of (V, f). We will define two families of trivectors of Type (D) whose base 3-spaces are not totally isotropic and five families of trivectors of Type (D) whose base 3-spaces are totally isotropic.

- We define

$$\gamma_1 \ := \ \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*.$$

- For every $\lambda \in \mathbb{F}^*$, we also define

$$\gamma_2(\lambda) \ := \ \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*.$$

Clearly, the base 3-space $\langle \bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^* \rangle$ of γ_1 and $\gamma_2(\lambda)$, $\lambda \in \mathbb{F}^*$, is not totally isotropic. We will prove the following.

Theorem 1.2 (1) *If χ is a trivector of Type (D) of V whose base 3-space is not totally isotropic with respect to f, then χ is $Sp(V, f)$-equivalent with γ_1 or $\gamma_2(\lambda)$ for some $\lambda \in \mathbb{F}^*$.*

(2) *For every $\lambda \in \mathbb{F}^*$, the trivectors γ_1 and $\gamma_2(\lambda)$ are not $Sp(V, f)$-equivalent.*

(3) *If $\lambda, \lambda' \in \mathbb{F}^*$, then the trivectors $\gamma_2(\lambda)$ and $\gamma_2(\lambda')$ are $Sp(V, f)$-equivalent if and only if $\lambda = \lambda'$.*

Any trivector of V which is $Sp(V, f)$-equivalent with γ_1 is called a *trivector of Type (D1)*. Any trivector of V which is $Sp(V, f)$-equivalent with a trivector of the form $\gamma_2(\lambda)$ for some $\lambda \in \mathbb{F}^*$ is called a *trivector of Type (D2)*.

- For all $\lambda_1, \lambda_2 \in \mathbb{F}^*$, we define

$$\begin{aligned} \gamma_3(\lambda_1, \lambda_2) \ &:= \ \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*, \\ \gamma_4(\lambda_1, \lambda_2) \ &:= \ \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*. \end{aligned}$$

Any trivector of V which is $Sp(V, f)$-equivalent with a trivector of the form $\gamma_3(\lambda_1, \lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{F}^*$ is called a *trivector of Type (D3)*. Any trivector of V which is $Sp(V, f)$-equivalent with a trivector of the form $\gamma_4(\lambda_1, \lambda_2)$ for some $\lambda_1, \lambda_2 \in \mathbb{F}^*$ is called a *trivector of Type (D4)*.

- For every $\lambda \in \mathbb{F}^*$, we define

$$\gamma_5(\lambda) \ := \ \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_2^* + \bar{f}_3^*) - \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*.$$

Any trivector of V which is $Sp(V, f)$-equivalent with a trivector of the form $\gamma_5(\lambda)$ for some $\lambda \in \mathbb{F}^*$ is called a *trivector of Type (D5)*.

• If $char(\mathbb{F}) \neq 2$, then we define the following additional trivector:

$$\gamma_6 \ := \ -\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*.$$

If $char(\mathbb{F}) \neq 2$, then any trivector of V which is $Sp(V,f)$-equivalent with γ_6 is called a *trivector of Type (D6)*.

• If $|\mathbb{F}| = 2$, then we define the following additional trivector:

$$\gamma_7 \ := \ \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*.$$

If $|\mathbb{F}| = 2$, then any trivector of V which is $Sp(V,f)$-equivalent with γ_7 is called a *trivector of Type (D7)*.

For every $i \in \{3,4,5,6,7\}$, the γ_i-trivector defined above is of Type (D) and has the totally isotropic 3-space $\langle \bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^* \rangle$ as base 3-space. We will prove the following.

Theorem 1.3 *The trivectors of Type (D) with totally isotropic base 3-spaces are precisely the trivectors of Type (D3), (D4), (D5), (D6) and (D7).*

If χ_1 is a trivector of Type (Di) with $i \in \{1,2\}$ and χ_2 is a trivector of Type (Dj) with $j \in \{3,4,5,6,7\}$, then χ_1 and χ_2 cannot be $Sp(V,f)$-equivalent since the base 3-space of χ_2 is totally isotropic while the base 3-space of χ_1 is not. The following theorem completes the classification of the $Sp(V,f)$-equivalence classes of trivectors of Type (D).

Theorem 1.4 (1) *Let $i, j \in \{3,4,5,6,7\}$ with $i \neq j$. Then no trivector of Type (Di) is $Sp(V,f)$-equivalent with a trivector of Type (Dj).*

(2) *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}^*$. Then the trivectors $\gamma_3(\lambda_1,\lambda_2)$ and $\gamma_3(\lambda_1',\lambda_2')$ are $Sp(V,f)$-equivalent if and only if the matrices $A = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_1\lambda_2} \end{bmatrix}$ and $A' = \begin{bmatrix} \frac{1}{\lambda_1'} & 0 & 0 \\ 0 & \frac{1}{\lambda_2'} & 0 \\ 0 & 0 & \frac{1}{\lambda_1'\lambda_2'} \end{bmatrix}$ are congruent, i.e. if and only if there exists a nonsingular (3×3)-matrix M over $\mathbb{F}$ such that $A' = M \cdot A \cdot M^T$.*

(3) *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}^*$. Then the trivectors $\gamma_4(\lambda_1,\lambda_2)$ and $\gamma_4(\lambda_1',\lambda_2')$ are $Sp(V,f)$-equivalent if and only if $\lambda_1 = \lambda_1'$ and there exist $s,t \in \mathbb{F}$ such that $t^2 + st\lambda_1 + s^2\lambda_1 = \frac{\lambda_2'}{\lambda_2}$.*

(4) *Let $\lambda, \lambda' \in \mathbb{F}^*$. If $char(\mathbb{F}) = 2$, then $\gamma_5(\lambda)$ and $\gamma_5(\lambda')$ are $Sp(V,f)$-equivalent if and only if $\frac{\lambda+\lambda'}{\lambda\lambda'}$ is of the form $\mu^2 + \mu$ for some $\mu \in \mathbb{F}$. If $char(\mathbb{F}) \neq 2$, then the trivectors $\gamma_5(\lambda)$ and $\gamma_5(\lambda')$ are always $Sp(V,f)$-equivalent.*

Theorem 1.2 which deals with the case where the base 3-space is not totally isotropic will be proved in Section 4. Theorems 1.3 and 1.4 which deal with the case where the base 3-space is totally isotropic will be proved in Section 5. Section 3 is devoted to proving some facts about trivectors of Type (D) which are valid regardless of whether the base 3-space is totally isotropic or not. One of the important tools we are going to use is a certain invariant map π which will be defined in Section 2.

2 The invariant map π

The following two lemmas are special cases of a more general result, see e.g. De Bruyn [2, Section 4].

Lemma 2.1 *For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f), let π_B denote the linear map from $\bigwedge^3 V$ to V defined by*

$$\pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \pi_B(\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \pi_B(\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{o},$$

$$\pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \pi_B(\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \pi_B(\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{o},$$
$$\pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \pi_B(\bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_1, \pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \pi_B(\bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{f}_1,$$
$$\pi_B(\bar{e}_2 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_2, \pi_B(\bar{f}_2 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{f}_2,$$
$$\pi_B(\bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{e}_3 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{e}_3, \pi_B(\bar{f}_3 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{f}_3 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{f}_3.$$

Then π_B is independent of the chosen hyperbolic basis B of (V, f).

Lemma 2.2 *For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f), let π'_B denote the linear map from $\bigwedge^4 V$ to $\bigwedge^2 V$ defined by*

$$\pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \bar{e}_2 \wedge \bar{e}_3,\ \pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \bar{e}_2 \wedge \bar{f}_3,$$

$$\pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \bar{f}_2 \wedge \bar{e}_3,\ \pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{f}_2 \wedge \bar{f}_3,$$
$$\pi'_B(\bar{e}_2 \wedge \bar{f}_2 \wedge \bar{e}_1 \wedge \bar{e}_3) = \bar{e}_1 \wedge \bar{e}_3,\ \pi'_B(\bar{e}_2 \wedge \bar{f}_2 \wedge \bar{e}_1 \wedge \bar{f}_3) = \bar{e}_1 \wedge \bar{f}_3,$$
$$\pi'_B(\bar{e}_2 \wedge \bar{f}_2 \wedge \bar{f}_1 \wedge \bar{e}_3) = \bar{f}_1 \wedge \bar{e}_3,\ \pi'_B(\bar{e}_2 \wedge \bar{f}_2 \wedge \bar{f}_1 \wedge \bar{f}_3) = \bar{f}_1 \wedge \bar{f}_3,$$
$$\pi'_B(\bar{e}_3 \wedge \bar{f}_3 \wedge \bar{e}_1 \wedge \bar{e}_2) = \bar{e}_1 \wedge \bar{e}_2,\ \pi'_B(\bar{e}_3 \wedge \bar{f}_3 \wedge \bar{e}_1 \wedge \bar{f}_2) = \bar{e}_1 \wedge \bar{f}_2,$$
$$\pi'_B(\bar{e}_3 \wedge \bar{f}_3 \wedge \bar{f}_1 \wedge \bar{e}_2) = \bar{f}_1 \wedge \bar{e}_2,\ \pi'_B(\bar{e}_3 \wedge \bar{f}_3 \wedge \bar{f}_1 \wedge \bar{f}_2) = \bar{f}_1 \wedge \bar{f}_2,$$
$$\pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \wedge \bar{f}_2, \qquad \pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_1 \wedge \bar{f}_1 + \bar{e}_3 \wedge \bar{f}_3,$$
$$\pi'_B(\bar{e}_2 \wedge \bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_3 \wedge \bar{f}_3.$$

Then π'_B is independent of the chosen hyperbolic basis B of (V, f).

Let $\pi : \bigwedge^3 V \cup \bigwedge^4 V \to V \cup \bigwedge^2 V$ be the map which sends α to $\pi_B(\alpha)$ if $\alpha \in \bigwedge^3 V$ and to $\pi'_B(\alpha)$ if $\alpha \in \bigwedge^4 V$. Here, B is some arbitrary hyperbolic basis of (V, f). Observe that by Lemmas 2.1 and 2.2, the map π is an invariant, that means, is independent of the considered hyperbolic basis B of (V, f).

3 General properties of trivectors of Type (D)

The following lemma states that the notion of base 3-space of a trivector of Type (D) is well-defined.

Lemma 3.1 *If $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ and $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ are two ordered bases of V such that $\gamma_B = \gamma_{B'}$, then $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle = \langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$.*

Proof. By De Bruyn [3, Section 7.3], the following holds:

- the subspace $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ consists of all vectors $\bar{x}$ of V for which $\bar{x} \wedge (\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2)$ is decomposable;
- the subspace $\langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$ consists of all vectors $\bar{x}$ of V for which $\bar{x} \wedge (\bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2')$ is decomposable.

Since $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2'$, we have $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle = \langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$. ■

If $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ is an ordered basis of V, then

(1) also $(\bar{v}_2, \bar{v}_1, \bar{v}_3, -\bar{w}_2, -\bar{w}_1, -\bar{w}_3)$ is an ordered basis of V;

(2) also $(\bar{v}_1, \bar{v}_3, \bar{v}_2, -\bar{w}_1, -\bar{w}_3, -\bar{w}_2)$ is an ordered basis of V;

(3) also $(\lambda\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \frac{\bar{w}_2}{\lambda}, \frac{\bar{w}_3}{\lambda})$, $\lambda \in \mathbb{F}^*$, is an ordered basis of V;

(4) also $(\bar{v}_1 + \bar{v}_2, \bar{v}_2, \bar{v}_3, \bar{w}_1 + \bar{w}_2, \bar{w}_2, \bar{w}_3)$ is an ordered basis of V;

(5) also $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1 + \lambda\bar{v}_2 + \mu\bar{v}_3, \bar{w}_2, \bar{w}_3)$, $\lambda, \mu \in \mathbb{F}$, is an ordered basis of V;

(6) also $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2 + \lambda\bar{v}_1 + \mu\bar{v}_3, \bar{w}_3)$, $\lambda, \mu \in \mathbb{F}$, is an ordered basis of V;

(7) also $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3 + \lambda\bar{v}_1 + \mu\bar{v}_2)$, $\lambda, \mu \in \mathbb{F}$, is an ordered basis of V;

(8) also $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1 + \lambda\bar{v}_1, \bar{w}_2 + \mu\bar{v}_2, \bar{w}_3 - (\lambda+\mu)\bar{v}_3)$, $\lambda, \mu \in \mathbb{F}$, is an ordered basis of V.

If B_1 and B_2 are two ordered bases of V and $i \in \{1, 2, \ldots, 8\}$, then we say that $(B_1, B_2) \in \Omega_i$ if B_2 can be obtained from B_1 as described in (i) above. If B_1 and B_2 are two ordered bases of V and $\emptyset \neq J \subseteq \{1, 2, \ldots, 8\}$, then we say that $(B_1, B_2) \in \Omega_J$ if there exist ordered bases $C_1, C_2, \ldots, C_k$ of V (for some $k \geq 1$) such that $C_1 = B_1$, $C_k = B_2$ and $(C_i, C_{i+1}) \in \bigcup_{j \in J} \Omega_j$ for every $i \in \{1, 2, \ldots, k-1\}$. We also define $\Omega := \Omega_{\{1,2,\ldots,8\}}$.

Lemma 3.2 *If $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ and $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ are two ordered bases of V, then $\gamma_B = \gamma_{B'}$ if and only if $(B, B') \in \Omega$. Moreover, if $\gamma_B = \gamma_{B'}$, then there exists an ordered bases B'' of V such that $(B, B'') \in \Omega_{\{1,2,3,4\}}$ and $(B'', B') \in \Omega_{\{5,6,7,8\}}$.*

Proof. It is straightforward to verify that if $(B, B') \in \Omega_i$ for some $i \in \{1, 2, \dots, 8\}$, then $\gamma_B = \gamma_{B'}$. Hence, if $(B, B') \in \Omega$, then also $\gamma_B = \gamma_{B'}$.

Conversely, suppose that

$$\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2'. \quad (1)$$

By Lemma 3.1, $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle = \langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$. If $\{\bar{g}_1, \bar{g}_2, \bar{g}_3\}$ is a basis of $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$, then since the linear isomorphisms of $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ defined by $(\bar{g}_1, \bar{g}_2, \bar{g}_3) \mapsto (\bar{g}_2, \bar{g}_1, \bar{g}_3)$, $(\bar{g}_1, \bar{g}_2, \bar{g}_3) \mapsto (\bar{g}_1, \bar{g}_3, \bar{g}_2)$, $(\bar{g}_1, \bar{g}_2, \bar{g}_3) \mapsto (\lambda \bar{g}_1, \bar{g}_2, \bar{g}_3)$ $(\lambda \in \mathbb{F}^*)$, $(\bar{g}_1, \bar{g}_2, \bar{g}_3) \mapsto (\bar{g}_1 + \bar{g}_2, \bar{g}_2, \bar{g}_3)$ generate $GL(\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle)$, there must exist an ordered basis $B'' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1'', \bar{w}_2'', \bar{w}_3'')$ of V such that $(B, B'') \in \Omega_{\{1,2,3,4\}}$ and

$$\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3'' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1'' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2''. \quad (2)$$

From equations (1) and (2), we obtain

$$\bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2' = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3'' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1'' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2''. \quad (3)$$

Hence, $\bar{v}_1' \wedge \bar{v}_2' \wedge \bar{v}_3' \wedge (\bar{w}_1' - \bar{w}_1'') = 0$, i.e. $\bar{w}_1' - \bar{w}_1'' \in \langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$. This implies that there exists an ordered basis $B''' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2''', \bar{w}_3''')$ such that $(B'', B''') \in \Omega_{\{5,8\}}$ and

$$\bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3'' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1'' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2'' = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3''' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2'''. \quad (4)$$

By equations (3) and (4), we find

$$\bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2' = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3''' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2'''. \quad (5)$$

This implies that $\bar{v}_1' \wedge \bar{v}_2' \wedge \bar{v}_3' \wedge (\bar{w}_3' - \bar{w}_3''') = 0$ and hence that $\bar{w}_3' - \bar{w}_3''' \in \langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$. So, there exists an ordered basis $B'''' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2'''', \bar{w}_3')$ of V such that $(B''', B'''') \in \Omega_{\{7,8\}}$ and

$$\bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3''' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2''' = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{w}_3' + \bar{v}_2' \wedge \bar{v}_3' \wedge \bar{w}_1' + \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2''''. \quad (6)$$

By equations (5) and (6), we find $\bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2' = \bar{v}_3' \wedge \bar{v}_1' \wedge \bar{w}_2''''$, i.e. $\bar{v}_3' \wedge \bar{v}_1' \wedge (\bar{w}_2' - \bar{w}_2'''') = 0$ and $\bar{w}_2' - \bar{w}_2'''' \in \langle \bar{v}_1', \bar{v}_3' \rangle$. This implies that $(B'''', B') \in \Omega_6$.

Since $(B, B'') \in \Omega_{\{1,2,3,4\}}$, $(B'', B''') \in \Omega_{\{5,8\}}$, $(B''', B'''') \in \Omega_{\{7,8\}}$ and $(B'''', B') \in \Omega_6$, we have $(B, B') \in \Omega$ and $(B'', B') \in \Omega_{\{5,6,7,8\}}$ as we needed to prove. ∎

Lemma 3.3 *Let* $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ *and* $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ *be two ordered bases of* V *such that* $\gamma_B = \gamma_{B'}$. *If* θ_1 *is the linear isomorphism of* $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle = \langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$ *mapping* $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ *to* $(\bar{v}_1', \bar{v}_2', \bar{v}_3')$ *and* θ_2 *is the linear isomorphism of* V *mapping* B *to* B', *then* $\det(\theta_2) = \det(\theta_1)^{-1}$.

Proof. (1) Suppose first that $(B, B') \in \Omega_i$ for some $i \in \{1, 2, \dots, 8\}$. If $i \in \{1, 2\}$, then $\det(\theta_1) = \det(\theta_2) = -1$. If $i = 3$, then $\det(\theta_1) = \lambda$ and $\det(\theta_2) = \lambda^{-1}$ for some $\lambda \in \mathbb{F}^*$. If $i \in \{4, 5, 6, 7, 8\}$, then $\det(\theta_1) = \det(\theta_2) = 1$. So, the claim of the lemma holds in this case.

(2) Consider next the most general case. Then we know that there exist ordered bases $C_1, C_2, \ldots, C_k$ of V (for some $k \in \mathbb{N} \setminus \{0\}$) such that $B = C_1$, $B' = C_k$ and $(C_i, C_{i+1}) \in \bigcup_{j \in \{1,2,\ldots,8\}} \Omega_j$ for every $i \in \{1, 2, \ldots, k-1\}$. By (1), we may suppose that $k \geq 3$. If $\theta_2^{(i)}$, $i \in \{1, 2, \ldots, k-1\}$, is the linear isomorphism of V mapping C_i to C_{i+1} and if $\theta_1^{(i)}$ is the linear isomorphism of $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ obtained by restricting $\theta_2^{(i)}$ to the subspace $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$, then we have $\theta_1 = \theta_1^{(k-1)} \circ \theta_1^{(k-2)} \circ \cdots \circ \theta_1^{(1)}$ and $\theta_2 = \theta_2^{(k-1)} \circ \theta_2^{(k-2)} \circ \cdots \circ \theta_2^{(1)}$. By part (1), we know that $\det(\theta_2^{(i)}) = \det(\theta_1^{(i)})^{-1}$ for every $i \in \{1, 2, \ldots, k-1\}$. Hence, $\det(\theta_2) = \prod_{i=1}^{k-1} \det(\theta_2^{(i)}) = \left(\prod_{i=1}^{k-1} \det(\theta_1^{(i)}) \right)^{-1} = \det(\theta_1)^{-1}$. ■

Lemma 3.4 *Let A be a nonsingular (3×3)-matrix over $\mathbb{F}$ and let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V. Put $[\bar{v}_1', \bar{v}_2', \bar{v}_3']^T = A \cdot [\bar{v}_1, \bar{v}_2, \bar{v}_3]^T$, $[\bar{w}_1', \bar{w}_2', \bar{w}_3']^T = \frac{1}{\det(A)} A \cdot [\bar{w}_1, \bar{w}_2, \bar{w}_3]^T$ and $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$. Then $\gamma_B = \gamma_{B'}$.*

Proof. The lemma can be proved by direct verification. However, we prefer to give an alternative way how one can see why the lemma is valid. One can easily see that if the lemma holds for two matrices A_1 and A_2, then it also holds for their product $A_1 A_2$. Now, $GL(3, \mathbb{F})$ is generated by the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (\lambda \in \mathbb{F}^*),$$

and the lemma is valid for each of these matrices (since $\gamma_B = \gamma_{B'}$ if $(B, B') \in \Omega_i$, $i \in \{1, 2, 3, 4\}$). ■

Lemma 3.5 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ and $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ be two ordered bases of V such that $\gamma_B = \gamma_{B'}$. Then the following holds.*

(1) *We have $(B, B') \in \Omega_{\{1,2,3,4\}}$ if and only if $[\bar{v}_1', \bar{v}_2', \bar{v}_3']^T = A \cdot [\bar{v}_1, \bar{v}_2, \bar{v}_3]^T$ and $[\bar{w}_1', \bar{w}_2', \bar{w}_3']^T = \frac{1}{\det(A)} A \cdot [\bar{w}_1, \bar{w}_2, \bar{w}_3]^T$ for some nonsingular (3×3)-matrix A over $\mathbb{F}$.*

(2) *We have $(B, B') \in \Omega_{\{5,6,7,8\}}$ if and only if $[\bar{v}_1', \bar{v}_2', \bar{v}_3']^T = [\bar{v}_1, \bar{v}_2, \bar{v}_3]^T$ and $[\bar{w}_1', \bar{w}_2', \bar{w}_3']^T = [\bar{w}_1, \bar{w}_2, \bar{w}_3]^T + C \cdot [\bar{v}_1, \bar{v}_2, \bar{v}_3]^T$ for some (3×3)-matrix C over $\mathbb{F}$ whose trace $Tr(C)$ is equal to 0.*

(3) *We have $\gamma_B = \gamma_{B'}$ if and only if $[\bar{v}_1', \bar{v}_2', \bar{v}_3']^T = A \cdot [\bar{v}_1, \bar{v}_2, \bar{v}_3]^T$ and $[\bar{w}_1', \bar{w}_2', \bar{w}_3']^T = \frac{1}{\det(A)} A \cdot [\bar{w}_1, \bar{w}_2, \bar{w}_3]^T + CA \cdot [\bar{v}_1, \bar{v}_2, \bar{v}_3]^T$ for some (3×3)-matrices A and C over $\mathbb{F}$ such that $\det(A) \neq 0$ and $Tr(C) = 0$.*

Proof. The proof of Claim (1) is already implicit in the above discussion, see the proof of Lemma 3.4. The verification of Claim (2) is straightforward. By Lemma 3.2, $\gamma_B = \gamma_{B'}$ if and only if there exists an ordered basis $B'' = (\bar{v}_1'', \bar{v}_2'', \bar{v}_3'', \bar{w}_1'', \bar{w}_2'', \bar{w}_3'')$ of V such that $(B, B'') \in \Omega_{\{1,2,3,4\}}$ and $(B'', B') \in \Omega_{\{5,6,7,8\}}$. Claim (3) then easily follows from Claims (1) and (2). ■

4 The case where the base 3-space is not totally isotropic

In this section, we determine all $Sp(V,f)$-equivalence classes of trivectors of Type (D) whose base 3-spaces are not totally isotropic.

Lemma 4.1 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is not totally isotropic. Then there exists an ordered basis $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ of V such that $\gamma_{B'} = \gamma_B$ and $\langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle \subset (\bar{v}_2')^{\perp_f}$.*

Proof. Let $\bar{v}_2'$ be a nonzero vector of $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \subset (\bar{v}_2')^{\perp_f}$ and let $\bar{v}_1', \bar{v}_3'$ be two vectors of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle = \langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$. By Lemma 3.4, there exist vectors $\bar{w}_1', \bar{w}_2', \bar{w}_3'$ of V such that $\gamma_{B'} = \gamma_B$, where $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$. ■

Lemma 4.2 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is not totally isotropic and $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \subset \bar{v}_2^{\perp_f}$. Then there exists an ordered basis $B' = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1', \bar{w}_2', \bar{w}_3')$ of V such that $\gamma_{B'} = \gamma_B$ and $\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle \subset \langle \bar{v}_1, \bar{v}_3 \rangle^{\perp_f}$.*

Proof. Since $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is not totally isotropic, we have that $f(\bar{v}_1, \bar{v}_3) \neq 0$. The lemma holds if we take

$$\bar{w}_1' := \bar{w}_1 + a_1\bar{v}_1 + b_1\bar{v}_3, \qquad \bar{w}_2' := \bar{w}_2 + a_2\bar{v}_1 - (a_1+b_3)\bar{v}_2 + b_2\bar{v}_3, \qquad \bar{w}_3' := \bar{w}_3 + a_3\bar{v}_1 + b_3\bar{v}_3,$$

where

$$a_i = \frac{f(\bar{v}_3, \bar{w}_i)}{f(\bar{v}_1, \bar{v}_3)}, \qquad b_i = -\frac{f(\bar{v}_1, \bar{w}_i)}{f(\bar{v}_1, \bar{v}_3)}$$

for every $i \in \{1, 2, 3\}$. ■

Lemma 4.3 *If $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ is an ordered basis of V such that $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \subset \langle \bar{v}_1, \bar{v}_3 \rangle^{\perp_f}$, then $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle^{\perp_f}$ is a one-dimensional subspace.*

Proof. If this is not the case, then $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ would be totally isotropic with respect to f and hence also $\langle \bar{v}_1, \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$. This is clearly impossible. ■

Lemma 4.4 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is not totally isotropic, $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \subset \bar{v}_2^{\perp_f}$, $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \subset \langle \bar{v}_1, \bar{v}_3 \rangle^{\perp_f}$ and $\bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{w}_1, \bar{w}_3 \rangle$. Then the trivector γ_B is $Sp(V,f)$-equivalent with $\gamma_2(\lambda)$ for some $\lambda \in \mathbb{F}^*$.*

Proof. We have $f(\bar{v}_2, \bar{w}_2) \neq 0$. Let B' be the ordered basis $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1', \bar{w}_2, \bar{w}_3')$ of V where

$$\bar{w}_1' := \bar{w}_1 - \frac{f(\bar{w}_1, \bar{w}_2)}{f(\bar{v}_2, \bar{w}_2)}\bar{v}_2, \qquad \bar{w}_3' := \bar{w}_3 + \frac{f(\bar{w}_2, \bar{w}_3)}{f(\bar{v}_2, \bar{w}_2)}\bar{v}_2.$$

Then $\gamma_{B'} = \gamma_B$, $\langle \bar{w}_1', \bar{w}_2, \bar{w}_3' \rangle \subset \langle \bar{v}_1, \bar{v}_3 \rangle^{\perp_f}$, $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \subset \langle \bar{w}_1', \bar{w}_3' \rangle^{\perp_f}$ and $\langle \bar{w}_1', \bar{w}_2, \bar{w}_3' \rangle \subset \bar{w}_2^{\perp_f}$. Now, we put $\bar{e}_1' := \bar{v}_1$, $\bar{e}_2' := \bar{v}_2$ and $\bar{e}_3' := \bar{w}_1'$. There exist unique vectors $\bar{f}_1' \in \langle \bar{v}_3 \rangle$, $\bar{f}_2' \in$

$\langle \bar{w}_2 \rangle$ and $\bar{f}_3' \in \langle \bar{w}_3' \rangle$ such that $f(\bar{e}_i', \bar{f}_i') = 1$ for every $i \in \{1,2,3\}$. Then $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ is a hyperbolic basis of (V,f) and $\gamma_B = \lambda_1 \cdot \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \lambda_2 \cdot \bar{e}_2' \wedge \bar{f}_1' \wedge \bar{e}_3' + \lambda_3 \cdot \bar{f}_1' \wedge \bar{e}_1' \wedge \bar{f}_2'$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}^*$. Now, put $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3) := (\frac{\bar{e}_1'}{\lambda_2\lambda_3}, \lambda_2\lambda_3\bar{f}_1', \frac{\bar{e}_2'}{\lambda_3}, \lambda_3\bar{f}_2', \bar{e}_3', \bar{f}_3')$. Then $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a hyperbolic basis of (V,f) and $\gamma_B = \lambda \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \bar{e}_2 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{e}_1 \wedge \bar{f}_2$ where $\lambda = \lambda_1\lambda_2\lambda_3^2$. So, γ_B is $Sp(V,f)$-equivalent with $\gamma_2(\lambda)$. ■

Lemma 4.5 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that*

- *$\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is not totally isotropic;*
- *$\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \subset \bar{v}_2^{\perp_f}$;*
- *$\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \subset \langle \bar{v}_1, \bar{v}_3 \rangle^{\perp_f}$;*
- *$\bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \neq \langle \bar{w}_1, \bar{w}_3 \rangle$.*

Then there exists an ordered basis $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ of V such that

- *$\gamma_{B'} = \gamma_B$;*
- *$\langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle$ is not totally isotropic;*
- *$\langle \bar{v}_1', \bar{v}_2', \bar{v}_3' \rangle \subset (\bar{v}_2')^{\perp_f}$;*
- *$\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle \subset \langle \bar{v}_1', \bar{v}_3' \rangle^{\perp_f}$;*
- *$\langle \bar{w}_1' \rangle = (\bar{v}_2')^{\perp_f} \cap \langle \bar{w}_1', \bar{w}_3' \rangle$;*
- *$\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle \cap \langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle^{\perp_f} \subset \langle \bar{w}_2', \bar{w}_3' \rangle$.*

Proof. Since $\bar{v}_2 \notin Rad(f)$, we have $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \not\subset \bar{v}_2^{\perp_f}$ and hence $\bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_3 \rangle = \langle \bar{u} \rangle$ for some nonzero vector $\bar{u} \in V$. We can take either $\bar{u} = \bar{w}_3$ or $\bar{u} = \bar{w}_1 + a\bar{w}_3$ for some $a \in \mathbb{F}$. Put $\langle \bar{w} \rangle := \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle^{\perp_f}$. Since $\bar{w} \notin Rad(f)$, we have $f(\bar{v}_2, \bar{w}) \neq 0$. If $\langle \bar{w} \rangle = \langle \bar{w}_2 \rangle$, then we take $\bar{v} := \bar{w}_1$. If $\langle \bar{w} \rangle \neq \langle \bar{w}_2 \rangle$, then let $\langle \bar{v} \rangle = \langle \bar{w}, \bar{w}_2 \rangle \cap \langle \bar{w}_1, \bar{w}_3 \rangle$. We can take either $\bar{v} = \bar{w}_1$ or $\bar{v} = \bar{w}_3 + b\bar{w}_1$ for some $b \in \mathbb{F}$. Clearly, $\langle \bar{w} \rangle \subset \langle \bar{v}, \bar{w}_2 \rangle$. We can distinguish the following cases.

(1) Suppose $\bar{u} = \bar{w}_3$ and $\bar{v} = \bar{w}_1$. Then the claims of the lemma hold if we take B' equal to $(\bar{v}_3, \bar{v}_2, \bar{v}_1, -\bar{w}_3, -\bar{w}_2, -\bar{w}_1)$.

(2) Suppose $\bar{u} = \bar{w}_3$ and $\bar{v} = \bar{w}_3$. Since $\bar{w} \in \langle \bar{w}_2, \bar{w}_3 \rangle$, we have $f(\bar{w}_2, \bar{w}_3) = 0$. Since $f(\bar{v}_2, \bar{w}_3) = 0$ and $f(\bar{v}_2, \bar{w}) \neq 0$, we have $f(\bar{v}_2, \bar{w}_2) \neq 0$. The claims of the lemma hold if we take B' equal to $(\bar{v}_3, \bar{v}_2, \bar{v}_1, -\bar{w}_3, -\bar{w}_2, -\bar{w}_1 + \frac{f(\bar{w}_2, \bar{w}_1)}{f(\bar{w}_2, \bar{v}_2)}\bar{v}_2)$. Observe that $\langle \bar{w}_3, \bar{w}_2, -\bar{w}_1 + \frac{f(\bar{w}_2, \bar{w}_1)}{f(\bar{w}_2, \bar{v}_2)}\bar{v}_2 \rangle \subset \bar{w}_2^{\perp_f}$.

(3) Suppose $\bar{u} = \bar{w}_3$ and $\bar{v} = \bar{w}_3 + b\bar{w}_1$ for some $b \in \mathbb{F}^*$. Then the claims of the lemma hold if we take B' equal to $(\bar{v}_3, \bar{v}_2, \bar{v}_3 + b\bar{v}_1, -\frac{1}{b}\bar{w}_3, -\frac{1}{b}\bar{w}_2, -(\frac{1}{b}\bar{w}_3 + \bar{w}_1))$.

(4) Suppose $\bar{u} = \bar{w}_1$ and $\bar{v} = \bar{w}_1$. Since $\bar{w} \in \langle \bar{w}_1, \bar{w}_2 \rangle$, we have $f(\bar{w}_1, \bar{w}_2) = 0$. Since $f(\bar{v}_2, \bar{w}_1) = 0$ and $f(\bar{v}_2, \bar{w}) \neq 0$, we have $f(\bar{v}_2, \bar{w}_2) \neq 0$. The claims of the lemma hold if we take B' equal to $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3 - \frac{f(\bar{w}_2, \bar{w}_3)}{f(\bar{w}_2, \bar{v}_2)}\bar{v}_2)$. Observe that $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 - \frac{f(\bar{w}_2, \bar{w}_3)}{f(\bar{w}_2, \bar{v}_2)}\bar{v}_2 \rangle \subset \bar{w}_2^{\perp_f}$.

(5) Suppose $\bar{u} = \bar{w}_1 + a\bar{w}_3$ and $\bar{v} = \bar{w}_1$ for some $a \in \mathbb{F}^*$. Then the claims of the lemma hold of we take B' equal to $(\bar{v}_1 + a\bar{v}_3, \bar{v}_2, \bar{v}_1, -(\frac{1}{a}\bar{w}_1 + \bar{w}_3), -\frac{1}{a}\bar{w}_2, -\frac{1}{a}\bar{w}_1)$.

(6) Suppose $\bar{u} = \bar{w}_1 + a\bar{w}_3$ and $\bar{v} = \bar{w}_3 + b\bar{w}_1$ where $ab \neq 1$. Then the claims of the lemma hold if we take B' equal to $(\bar{v}_1 + a\bar{v}_3, \bar{v}_2, \bar{v}_3 + b\bar{v}_1, \frac{1}{1-ab}(\bar{w}_1 + a\bar{w}_3), \frac{1}{1-ab}\bar{w}_2, \frac{1}{1-ab}(\bar{w}_3 + b\bar{w}_1))$.

(7) Suppose $\bar{u} = \bar{w}_1 + a\bar{w}_3$ and $\bar{v} = \bar{w}_3 + b\bar{w}_1$ where $ab = 1$. Since $\bar{w} \in \langle \bar{w}_2, \bar{w}_3 + b\bar{w}_1 \rangle = \langle \bar{w}_2, \bar{w}_1 + a\bar{w}_3 \rangle$, we have $f(\bar{w}_1 + a\bar{w}_3, \bar{w}_2) = 0$. Since $f(\bar{v}_2, \bar{w}_1 + a\bar{w}_3) = f(\bar{v}_2, \bar{w}_3 + b\bar{w}_1) = 0$ and $f(\bar{v}_2, \bar{w}) \neq 0$, we have $f(\bar{v}_2, \bar{w}_2) \neq 0$. The claims of the lemma hold if we take B' equal to $(\bar{v}_1 + a\bar{v}_3, \bar{v}_2, \bar{v}_3, \bar{w}_1 + a\bar{w}_3, \bar{w}_2, \bar{w}_3 - \frac{f(\bar{w}_2, \bar{w}_3)}{f(\bar{w}_2, \bar{v}_2)}\bar{v}_2)$. Observe that $\langle \bar{w}_1 + a\bar{w}_3, \bar{w}_2, \bar{w}_3 - \frac{f(\bar{w}_2, \bar{w}_3)}{f(\bar{w}_2, \bar{v}_2)}\bar{v}_2 \rangle \subset \bar{w}_2^{\perp_f}$. ∎

Lemma 4.6 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that*

- $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ *is not totally isotropic;*
- $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \subset \bar{v}_2^{\perp_f}$*;*
- $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \subset \langle \bar{v}_1, \bar{v}_3 \rangle^{\perp_f}$*;*
- $\langle \bar{w}_1 \rangle = \bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_3 \rangle$*;*
- $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle^{\perp_f} \subset \langle \bar{w}_2, \bar{w}_3 \rangle$.

Then there exists an ordered basis $B' = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1', \bar{w}_2, \bar{w}_3)$ of V such that

- $\gamma_{B'} = \gamma_B$*;*
- $\langle \bar{w}_1', \bar{w}_2, \bar{w}_3 \rangle \subset \langle \bar{v}_1, \bar{v}_3 \rangle^{\perp_f}$*;*
- $\langle \bar{w}_1' \rangle = \bar{v}_2^{\perp_f} \cap \langle \bar{w}_1', \bar{w}_3 \rangle$*;*
- $\langle \bar{w}_1', \bar{w}_2, \bar{w}_3 \rangle \cap \langle \bar{w}_1', \bar{w}_2, \bar{w}_3 \rangle^{\perp_f} = \langle \bar{w}_3 \rangle$.

Proof. If $\bar{x} \in (\bar{v}_2^{\perp_f} \cap \langle \bar{w}_2, \bar{w}_3 \rangle) \cap (\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle^{\perp_f})$, then $\bar{x}^{\perp_f} = V$ and hence $\bar{x} = 0$. So, there exist unique vectors $\bar{u} \in \bar{v}_2^{\perp_f} \cap \langle \bar{w}_2, \bar{w}_3 \rangle$ and $\bar{w} \in \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle^{\perp_f}$ such that $\langle \bar{u} \rangle \neq \langle \bar{w} \rangle$ and $f(\bar{w}_1, \bar{u}) = f(\bar{v}_2, \bar{w}) = 1$. Since $\bar{u}, \bar{w} \in \langle \bar{w}_2, \bar{w}_3 \rangle$ with $\langle \bar{u} \rangle \neq \langle \bar{w} \rangle$, there exist unique $a, b, c, d \in \mathbb{F}$ such that $\bar{w}_2 = a\bar{u} + b\bar{w}$ and $\bar{w}_3 = c\bar{u} + d\bar{w}$. Since $\bar{w}_2$ and $\bar{w}_3$ are linearly independent, we have $ad - bc \neq 0$. Since $\bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_3 \rangle = \langle \bar{w}_1 \rangle$ and $\bar{u} \in \bar{v}_2^{\perp_f}$, we have $d \neq 0$. We also have that $f(\bar{v}_2, \bar{w}_2) = b \cdot f(\bar{v}_2, \bar{w}) = b$, $f(\bar{v}_2, \bar{w}_3) = d \cdot f(\bar{v}_2, \bar{w}) = d$, $f(\bar{w}_1, \bar{w}_2) = a \cdot f(\bar{w}_1, \bar{u}) = a$, $f(\bar{w}_1, \bar{w}_3) = c \cdot f(\bar{w}_1, \bar{u}) = c$ and $f(\bar{w}_2, \bar{w}_3) = (ad - bc) \cdot f(\bar{u}, \bar{w}) = 0$.

Now, put $\bar{w}_1' := \bar{w}_1 - \frac{c}{d}\bar{v}_2$. Since $f(\bar{w}_1', \bar{w}_3) = f(\bar{w}_1 - \frac{c}{d}\bar{v}_2, \bar{w}_3) = f(\bar{w}_1, \bar{w}_3) - \frac{c}{d} f(\bar{v}_2, \bar{w}_3) = c - \frac{c}{d} \cdot d = 0$ and $f(\bar{w}_2, \bar{w}_3) = 0$, we have $\langle \bar{w}_1', \bar{w}_2, \bar{w}_3 \rangle \subseteq \bar{w}_3^{\perp_f}$. The hyperbolic basis $B' = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1', \bar{w}_2, \bar{w}_3)$ now satisfies the required conditions. ∎

For every $\lambda \in \mathbb{F}$, let $\gamma_1(\lambda)$ be the trivector

$$\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge (\bar{f}_3^* + \lambda \bar{f}_2^*),$$

where $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ is the fixed hyperbolic basis of (V, f) considered in Section 1. Observe that $\gamma_1(0) = \gamma_1$.

Lemma 4.7 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that*

- $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ *is not totally isotropic;*
- $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \subset \bar{v}_2^{\perp_f}$*;*

- $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \subset \langle \bar{v}_1, \bar{v}_3 \rangle^{\perp_f}$;
- $\langle \bar{w}_1 \rangle = \bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_3 \rangle$;
- $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle^{\perp_f} = \langle \bar{w}_3 \rangle$.

Then the trivector γ_B is $Sp(V, f)$-equivalent with the trivector $\gamma_1(\lambda)$ for some $\lambda \in \mathbb{F}$.

Proof. Since $\bar{w}_3 \notin Rad(f)$, we have $f(\bar{v}_2, \bar{w}_3) \neq 0$. Put $\bar{v}_2^{\perp_f} \cap \langle \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{u} \rangle$. We may suppose that $\bar{u} = \bar{w}_2 + \lambda' \bar{w}_3$ where λ' is some element of $\mathbb{F}$.

We put $\bar{e}_1' := \bar{v}_1$, $\bar{e}_2' := \bar{v}_2$ and $\bar{e}_3' := \bar{w}_1$. Then there exist unique vectors $\bar{f}_1' \in \langle \bar{v}_3 \rangle$, $\bar{f}_2' \in \langle \bar{w}_3 \rangle$, $\bar{f}_3' \in \langle \bar{u} \rangle$ such that $f(\bar{e}_i', \bar{f}_i') = 1$ for every $i \in \{1, 2, 3\}$. Then $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ is a hyperbolic basis of (V, f) and $\gamma_B = \lambda_1 \cdot \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_2' + \lambda_2 \cdot \bar{e}_2' \wedge \bar{f}_1' \wedge \bar{e}_3' + \lambda_3 \cdot \bar{f}_1' \wedge \bar{e}_1' \wedge (\bar{f}_3' + \lambda_4 \bar{f}_2')$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}^*$ and some $\lambda_4 \in \mathbb{F}$. If we put $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3) = (\lambda_1 \bar{e}_1', \frac{\bar{f}_1'}{\lambda_1}, \lambda_1 \lambda_2 \lambda_3 \bar{e}_2', \frac{\bar{f}_2'}{\lambda_1 \lambda_2 \lambda_3}, \frac{\bar{e}_3'}{\lambda_3}, \lambda_3 \bar{f}_3')$, then $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a hyperbolic basis of (V, f) and $\gamma_B = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_2 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{e}_1 \wedge (\bar{f}_3 + \lambda \bar{f}_2)$ where $\lambda = \lambda_1 \lambda_2 \lambda_3^2 \lambda_4$. ∎

Lemma 4.8 *For every $\lambda \in \mathbb{F}^*$, the trivectors $\gamma_1(\lambda)$ and $\gamma_2(-\lambda)$ are $Sp(V, f)$-equivalent.*

Proof. We have

$$\gamma_1(\lambda) = (-\lambda) \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \bar{e}_2 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{e}_1 \wedge \bar{f}_2,$$

where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(-\bar{e}_1^*, -\bar{f}_1^* + \frac{\bar{e}_2^*}{\lambda}, \frac{\bar{e}_2^*}{\lambda}, -\bar{e}_1^* + \bar{f}_3^* + \lambda \bar{f}_2^*, -\lambda \bar{e}_3^* + \bar{e}_2^*, -\frac{\bar{f}_3^*}{\lambda})$ of (V, f). ∎

The following lemma is precisely Theorem 1.2(1).

Lemma 4.9 *If χ is a trivector of Type (D) of V whose base 3-space is not totally isotropic with respect to f, then χ is $Sp(V, f)$-equivalent with γ_1 or $\gamma_2(\lambda)$ for some $\lambda \in \mathbb{F}^*$.*

Proof. This is a consequence of Lemmas 4.1, 4.2, 4.4, 4.5, 4.6, 4.7 and 4.8. ∎

The following lemma is precisely Theorem 1.2(2).

Lemma 4.10 *For every $\lambda \in \mathbb{F}^*$, the trivectors γ_1 and $\gamma_2(\lambda)$ are not $Sp(V, f)$-equivalent.*

Proof. Suppose the trivectors γ_1 and $\gamma_2(\lambda)$ are $Sp(V, f)$-equivalent. Then there exists a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) such that $\gamma_2(\lambda) = \gamma := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + \bar{e}_2 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{e}_1 \wedge \bar{f}_3$. Since $\gamma_2(\lambda) = \gamma$, we must have $\pi(\gamma_2(\lambda)) \wedge \pi[\gamma_2(\lambda) \wedge \pi(\gamma_2(\lambda))] = \pi(\gamma) \wedge \pi[\gamma \wedge \pi(\gamma)]$.

We have $\pi(\gamma_2(\lambda)) = -\bar{f}_2^*$. Hence, $\gamma_2(\lambda) \wedge \pi(\gamma_2(\lambda)) = \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* - \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* \wedge \bar{e}_3^*$, $\pi[\gamma_2(\lambda) \wedge \pi(\gamma_2(\lambda))] = \lambda \cdot \bar{e}_1^* \wedge \bar{f}_3^* - \bar{f}_1^* \wedge \bar{e}_3^*$ and $\pi(\gamma_2(\lambda)) \wedge \pi[\gamma_2(\lambda) \wedge \pi(\gamma_2(\lambda))] = \lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* - \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^*$.

On the other hand, we have $\pi(\gamma) = \bar{e}_1 - \bar{f}_3$, $\gamma \wedge \pi(\gamma) = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 - \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 \wedge \bar{f}_3 + \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3$, $\pi(\gamma \wedge \pi(\gamma)) = \bar{e}_2 \wedge \bar{e}_3 - \bar{e}_1 \wedge \bar{f}_3 + \bar{f}_1 \wedge \bar{e}_2$ and $\pi(\gamma) \wedge \pi(\gamma \wedge \pi(\gamma)) = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 - \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3 - \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 = (\bar{e}_1 - \bar{f}_3) \wedge (\bar{e}_2 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{e}_2) = (\bar{e}_1 - \bar{f}_3) \wedge (\bar{f}_1 - \bar{e}_3) \wedge \bar{e}_2$.

The equality $\pi(\gamma_2(\lambda)) \wedge \pi[\gamma_2(\lambda) \wedge \pi(\gamma_2(\lambda))] = \pi(\gamma) \wedge \pi[\gamma \wedge \pi(\gamma)]$ implies that $\lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* - \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* = (\bar{e}_1 - \bar{f}_3) \wedge (\bar{f}_1 - \bar{e}_3) \wedge \bar{e}_2$. This is impossible since the former trivector is of Type (B), while the latter trivector is of Type (A). ∎

The proof of Theorem 1.2(3) will rely on two lemmas.

Lemma 4.11 *Let $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ and $\{\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{v}_4'\}$ be two linearly independent sets of vectors of V. If $\bar{v}_1 \wedge \bar{v}_2 + \bar{v}_3 \wedge \bar{v}_4 = \bar{v}_1' \wedge \bar{v}_2' + \bar{v}_3' \wedge \bar{v}_4'$, then $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4 \rangle = \langle \bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{v}_4' \rangle$.*

Proof. Let $\bar{x} \in V$. We prove that $(\bar{v}_1 \wedge \bar{v}_2 + \bar{v}_3 \wedge \bar{v}_4) \wedge \bar{x}$ is decomposable if and only if $\bar{x} \in \langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4 \rangle$. If $\bar{x} \notin \langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4 \rangle$, then $(\bar{v}_1 \wedge \bar{v}_2 + \bar{v}_3 \wedge \bar{v}_4) \wedge \bar{x}$ is a trivector of Type (B) and hence indecomposable. If $\bar{x} \in \langle \bar{v}_1, \bar{v}_2 \rangle \cup \langle \bar{v}_3, \bar{v}_4 \rangle$, then clearly $(\bar{v}_1 \wedge \bar{v}_2 + \bar{v}_3 \wedge \bar{v}_4) \wedge \bar{x}$ is decomposable. Suppose now that $\bar{x} \in \langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4 \rangle \setminus (\langle \bar{v}_1, \bar{v}_2 \rangle \cup \langle \bar{v}_3, \bar{v}_4 \rangle)$. Then $\bar{x} = \bar{v}_2' + \bar{v}_4'$ for some nonzero $\bar{v}_2' \in \langle \bar{v}_1, \bar{v}_2 \rangle$ and some nonzero $\bar{v}_4' \in \langle \bar{v}_3, \bar{v}_4 \rangle$. Let $\bar{v}_1' \in \langle \bar{v}_1, \bar{v}_2 \rangle$ and $\bar{v}_3' \in \langle \bar{v}_3, \bar{v}_4 \rangle$ such that $\bar{v}_1' \wedge \bar{v}_2' = \bar{v}_1 \wedge \bar{v}_2$ and $\bar{v}_3' \wedge \bar{v}_4' = \bar{v}_3 \wedge \bar{v}_4$. Then $(\bar{v}_1 \wedge \bar{v}_2 + \bar{v}_3 \wedge \bar{v}_4) \wedge \bar{x} = (\bar{v}_1' \wedge \bar{v}_2' + \bar{v}_3' \wedge \bar{v}_4') \wedge (\bar{v}_2' + \bar{v}_4') = \bar{v}_1' \wedge \bar{v}_2' \wedge \bar{v}_4' + \bar{v}_3' \wedge \bar{v}_4' \wedge \bar{v}_2' = (\bar{v}_1' - \bar{v}_3') \wedge \bar{v}_2' \wedge \bar{v}_4'$ is decomposable.

By the previous paragraph, we know that the set of all $\bar{x} \in V$ for which $(\bar{v}_1 \wedge \bar{v}_2 + \bar{v}_3 \wedge \bar{v}_4) \wedge \bar{x} = (\bar{v}_1' \wedge \bar{v}_2' + \bar{v}_3' \wedge \bar{v}_4') \wedge \bar{x}$ is decomposable is equal to $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4 \rangle = \langle \bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{v}_4' \rangle$.
∎

The following lemma is precisely Lemma 2.9 of De Bruyn and Kwiatkowski [4].

Lemma 4.12 *Let U be a 4-dimensional vector space over the field $\mathbb{F}$ and suppose $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4\}$ and $\{\bar{u}_1', \bar{u}_2', \bar{u}_3', \bar{u}_4'\}$ are two bases of U such that $\bar{u}_1 \wedge \bar{u}_2 + \bar{u}_3 \wedge \bar{u}_4 = \bar{u}_1' \wedge \bar{u}_2' + \bar{u}_3' \wedge \bar{u}_4'$. Then $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 \wedge \bar{u}_4 = \bar{u}_1' \wedge \bar{u}_2' \wedge \bar{u}_3' \wedge \bar{u}_4'$.*

The following lemma is precisely Theorem 1.2(3).

Lemma 4.13 *If $\lambda, \lambda' \in \mathbb{F}^*$, then the trivectors $\gamma_2(\lambda)$ and $\gamma_2(\lambda')$ are $Sp(V, f)$-equivalent if and only if $\lambda = \lambda'$.*

Proof. Suppose $\gamma_2(\lambda)$ and $\gamma_2(\lambda')$ are $Sp(V, f)$-equivalent. Then there exists a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) such that

$$\lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_2^* = \gamma := \lambda' \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \bar{e}_2 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_1 \wedge \bar{e}_1 \wedge \bar{f}_2.$$

We must have that $\pi[\gamma_2(\lambda) \wedge \pi(\gamma_2(\lambda))] = \pi[\gamma \wedge \pi(\gamma)]$. By the calculations made in Lemma 4.10, this implies that

$$\lambda \cdot \bar{e}_1^* \wedge \bar{f}_3^* - \bar{f}_1^* \wedge \bar{e}_3^* = \lambda' \cdot \bar{e}_1 \wedge \bar{f}_3 - \bar{f}_1 \wedge \bar{e}_3.$$

By Lemmas 4.11 and 4.12, we then have $U := \langle \bar{e}_1^*, \bar{f}_1^*, \bar{e}_3^*, \bar{f}_3^* \rangle = \langle \bar{e}_1, \bar{f}_1, \bar{e}_3, \bar{f}_3 \rangle$ and $\lambda \cdot \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* = \lambda' \cdot \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3$. Since $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_3^*, \bar{f}_3^*)$ and $(\bar{e}_1, \bar{f}_1, \bar{e}_3, \bar{f}_3)$ are two hyperbolic bases of $(U, f_{|U})$, we have $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3$ and hence $\lambda = \lambda'$. ∎

5 The case where the base 3-space is totally isotropic

In this section, we determine all $Sp(V,f)$-equivalence classes of trivectors of Type (D) whose base 3-spaces are totally isotropic.

Lemma 5.1 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ and $B' = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1', \bar{w}_2', \bar{w}_3')$ be two ordered bases of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$, $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ and $\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle$ are totally isotropic. Then the following holds.*

(1) If $f(\bar{v}_i, \bar{w}_j) = f(\bar{v}_i, \bar{w}_j')$ for all $i, j \in \{1,2,3\}$, then the trivectors γ_B and $\gamma_{B'}$ are $Sp(V,f)$-equivalent.

(2) If $\gamma_B = \gamma_{B'}$, then $f(\bar{v}_i, \bar{w}_j) = f(\bar{v}_i, \bar{w}_j')$ for all $i, j \in \{1,2,3\}$.

Proof. (1) Suppose $f(\bar{v}_i, \bar{w}_j) = f(\bar{v}_i, \bar{w}_j')$ for all $i, j \in \{1,2,3\}$. Since $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ and $\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle$ are totally isotropic, there exist hyperbolic bases $(\bar{v}_1, \bar{f}_1, \bar{v}_2, \bar{f}_2, \bar{v}_3, \bar{f}_3)$ and $(\bar{v}_1, \bar{f}_1', \bar{v}_2, \bar{f}_2', \bar{v}_3, \bar{f}_3')$ of (V,f) such that $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{f}_1, \bar{f}_2, \bar{f}_3 \rangle$ and $\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle = \langle \bar{f}_1', \bar{f}_2', \bar{f}_3' \rangle$. Let a_{ij} and a_{ij}' $(i,j \in \{1,2,3\})$ be the unique elements of $\mathbb{F}$ such that $\bar{w}_i = a_{i1}\bar{f}_1 + a_{i2}\bar{f}_2 + a_{i3}\bar{f}_3$ and $\bar{w}_i' = a_{i1}'\bar{f}_1' + a_{i2}'\bar{f}_2' + a_{i3}'\bar{f}_3'$ for every $i \in \{1,2,3\}$. Since $f(\bar{v}_j, \bar{w}_i) = f(\bar{v}_j, \bar{w}_i')$, we have $a_{ij} = a_{ij}'$ for all $i, j \in \{1,2,3\}$. So, if θ is the element of $Sp(V,f)$ mapping $(\bar{v}_1, \bar{f}_1, \bar{v}_2, \bar{f}_2, \bar{v}_3, \bar{f}_3)$ to $(\bar{v}_1, \bar{f}_1', \bar{v}_2, \bar{f}_2', \bar{v}_3, \bar{f}_3')$, then $\bigwedge^3(\theta)(\gamma_B) = \gamma_{B'}$. So, γ_B and γ_B' are $Sp(V,f)$-equivalent.

(2) Suppose $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3' + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1' + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2'$ and let $i, j \in \{1,2,3\}$. If we consider the wedge product of both sides of the equality with the vector $\bar{v}_j$, then we find that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_j = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_j'$, i.e. $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 \wedge (\bar{w}_j' - \bar{w}_j) = 0$. This implies that $\bar{w}_j' - \bar{w}_j \in \langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ and hence that $f(\bar{v}_i, \bar{w}_j) = f(\bar{v}_i, \bar{w}_j')$ since $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is totally isotropic. ∎

Lemma 5.2 *Let $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ be a hyperbolic basis of (V,f) and let M be a nonsingular (3×3)-matrix over $\mathbb{F}$. Put $[\bar{e}_1', \bar{e}_2', \bar{e}_3']^T := M \cdot [\bar{e}_1, \bar{e}_2, \bar{e}_3]^T$ and $[\bar{f}_1', \bar{f}_2', \bar{f}_3']^T := (M^T)^{-1} \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T$. Then $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ is a hyperbolic basis of (V,f).*

Proof. Put $N := (M^T)^{-1} = (M^{-1})^T$ and let $i, j \in \{1,2,3\}$. Then we have $f(\bar{e}_i', \bar{f}_j') = f(\sum_{k=1}^3 M_{ik}\bar{e}_k, \sum_{l=1}^3 N_{jl}\bar{f}_l) = \sum_{k=1}^3 \sum_{l=1}^3 M_{ik}N_{jl} \cdot f(\bar{e}_k, \bar{f}_l) = \sum_{k=1}^3 M_{ik}N_{jk} = (M \cdot N^T)_{ij} = \delta_{ij}$. ∎

Lemma 5.3 *Let $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ and $B' = (\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ be two hyperbolic bases of V such that $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle = \langle \bar{e}_1', \bar{e}_2', \bar{e}_3' \rangle$. Let A and A' be two nonsingular (3×3)-matrices over $\mathbb{F}$. Let M be the (3×3)-matrix over $\mathbb{F}$ such that $[\bar{e}_1', \bar{e}_2', \bar{e}_3']^T = M \cdot [\bar{e}_1, \bar{e}_2, \bar{e}_3]^T$ and put $[\bar{w}_1, \bar{w}_2, \bar{w}_3]^T := A \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T$ and $[\bar{w}_1', \bar{w}_2', \bar{w}_3']^T := A' \cdot [\bar{f}_1', \bar{f}_2', \bar{f}_3']^T$. If the trivectors $\alpha = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{w}_3 + \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{w}_1 + \bar{e}_3 \wedge \bar{e}_1 \wedge \bar{w}_2$ and $\alpha' = \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{w}_3' + \bar{e}_2' \wedge \bar{e}_3' \wedge \bar{w}_1' + \bar{e}_3' \wedge \bar{e}_1' \wedge \bar{w}_2'$ are equal, then $det(M) \cdot A' = MAM^T$.*

Proof. Put $[\bar{f}_1'', \bar{f}_2'', \bar{f}_3'']^T = (M^T)^{-1} \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T$ and $[\bar{w}_1'', \bar{w}_2'', \bar{w}_3''] := \frac{1}{\det(M)} M \cdot [\bar{w}_1, \bar{w}_2, \bar{w}_3]^T$ $= \frac{1}{\det(M)} MA \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T = \frac{1}{\det(M)} MAM^T \cdot [\bar{f}_1'', \bar{f}_2'', \bar{f}_3'']^T$. Now, $(\bar{e}_1', \bar{f}_1'', \bar{e}_2', \bar{f}_2'', \bar{e}_3', \bar{f}_3'')$ is a hyperbolic basis of (V,f) by Lemma 5.2 and $\alpha = \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{w}_3'' + \bar{e}_2' \wedge \bar{e}_3' \wedge \bar{w}_1'' + \bar{e}_3' \wedge \bar{e}_1' \wedge \bar{w}_2''$ by Lemma 3.4. Since $\alpha = \alpha'$, Lemma 5.1(2) implies that $f(\bar{e}_i', \bar{w}_j'') = f(\bar{e}_i', \bar{w}_j')$ for all $i, j \in \{1,2,3\}$. This implies that $A' = \frac{1}{\det(M)} \cdot MAM^T$. ∎

Lemma 5.4 *Let* $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ *be a hyperbolic basis of* (V, f) *and let* A, A' *be two nonsingular* (3×3)*-matrices over* $\mathbb{F}$*. Put* $B := (\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ *and* $B' := (\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{w}'_1, \bar{w}'_2, \bar{w}'_3)$*, where* $[\bar{w}_1, \bar{w}_2, \bar{w}_3]^T := A \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T$ *and* $[\bar{w}'_1, \bar{w}'_2, \bar{w}'_3]^T := A' \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T$*. Then* γ_B *and* $\gamma_{B'}$ *are* $Sp(V, f)$*-equivalent if and only if there exists a nonsingular* (3×3)*-matrix* M *over* $\mathbb{F}$ *such that* $\det(M) \cdot A' = MAM^T$*. If* M *is a nonsingular* (3×3)*-matrix over* $\mathbb{F}$ *such that* $\det(M) \cdot A' = MAM^T$*, then* $\det(M) = \frac{\det(A)}{\det(A')}$*.*

Proof. Suppose the trivectors γ_B and $\gamma_{B'}$ are $Sp(V, f)$-equivalent. Then there exists a hyperbolic basis $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ of (V, f) such that $\gamma_{B'} = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{w}''_3 + \bar{e}'_2 \wedge \bar{e}'_3 \wedge \bar{w}''_1 + \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{w}''_2$, where $[\bar{w}''_1, \bar{w}''_2, \bar{w}''_3]^T := A \cdot [\bar{f}'_1, \bar{f}'_2, \bar{f}'_3]^T$. Indeed, we can take $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3) = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)^\theta$ where $\theta \in Sp(V, f)$ such that $\bigwedge^3(\theta)(\gamma_B) = \gamma_{B'}$. By Lemma 3.1, $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle = \langle \bar{e}'_1, \bar{e}'_2, \bar{e}'_3 \rangle$. Let M be the nonsingular (3×3)-matrix over $\mathbb{F}$ such that $[\bar{e}_1, \bar{e}_2, \bar{e}_3]^T := M \cdot [\bar{e}'_1, \bar{e}'_2, \bar{e}'_3]^T$. By Lemma 5.3, $\det(M) \cdot A' = MAM^T$.

Conversely, suppose that M is a nonsingular (3×3)-matrix over $\mathbb{F}$ such that $\det(M) \cdot A' = MAM^T$. Put $[\bar{e}'_1, \bar{e}'_2, \bar{e}'_3]^T := M^{-1} \cdot [\bar{e}_1, \bar{e}_2, \bar{e}_3]^T$ and $[\bar{f}'_1, \bar{f}'_2, \bar{f}'_3]^T := M^T \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T$. Then $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ is a hyperbolic basis of (V, f) by Lemma 5.2. Let θ be the element of $Sp(V, f)$ mapping $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ to $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$. Then $\bigwedge^3(\theta)(\gamma_B) = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{w}''_3 + \bar{e}'_2 \wedge \bar{e}'_3 \wedge \bar{w}''_1 + \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{w}''_2$, where $[\bar{w}''_1, \bar{w}''_2, \bar{w}''_3]^T = A \cdot [\bar{f}'_1, \bar{f}'_2, \bar{f}'_3]^T$. Now, put $[\bar{w}'''_1, \bar{w}'''_2, \bar{w}'''_3]^T := \frac{1}{\det(M^{-1})} \cdot M^{-1} \cdot [\bar{w}'_1, \bar{w}'_2, \bar{w}'_3]^T = \frac{1}{\det(M^{-1})} \cdot M^{-1} A' \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T = \frac{1}{\det(M^{-1})} \cdot M^{-1} A' (M^T)^{-1} \cdot [\bar{f}'_1, \bar{f}'_2, \bar{f}'_3]^T$. Then $\gamma_{B'} = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{w}'_3 + \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{w}'_1 + \bar{e}_3 \wedge \bar{e}_1 \wedge \bar{w}'_2 = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{w}'''_3 + \bar{e}'_2 \wedge \bar{e}'_3 \wedge \bar{w}'''_1 + \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{w}'''_2$ by Lemma 3.4. Since $\frac{1}{\det(M^{-1})} M^{-1} A' (M^T)^{-1} = A$, we have $f(\bar{e}'_i, \bar{w}''_j) = f(\bar{e}'_i, \bar{w}'''_j)$ for all $i, j \in \{1, 2, 3\}$. Lemma 5.1(1) then implies that the trivectors $\bigwedge^3(\theta)(\gamma_B)$ and $\gamma_{B'}$ are $Sp(V, f)$-equivalent. Hence, also the trivectors γ_B and $\gamma_{B'}$ are $Sp(V, f)$-equivalent.

Suppose M is a nonsingular (3×3)-matrix over $\mathbb{F}$ such that $\det(M) \cdot A' = MAM^T$. Then $\det(M)^3 \cdot \det(A') = \det(M) \cdot \det(A) \cdot \det(M)$ and hence $\det(M) = \frac{\det(A)}{\det(A')}$. ∎

Corollary 5.5 *Let* $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ *be a hyperbolic basis of* (V, f) *and let* A, A' *be two nonsingular* (3×3)*-matrices over* $\mathbb{F}$*. Put* $B := (\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ *and* $B' := (\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{w}'_1, \bar{w}'_2, \bar{w}'_3)$*, where* $[\bar{w}_1, \bar{w}_2, \bar{w}_3]^T := A \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T$ *and* $[\bar{w}'_1, \bar{w}'_2, \bar{w}'_3]^T := A' \cdot [\bar{f}_1, \bar{f}_2, \bar{f}_3]^T$*. Then* γ_B *and* $\gamma_{B'}$ *are* $Sp(V, f)$*-equivalent if and only if the matrices* $\frac{A}{\det(A)}$ *and* $\frac{A'}{\det(A')}$ *are congruent.*

Proof. Suppose γ_B and $\gamma_{B'}$ are $Sp(V, f)$-equivalent. Then by Lemma 5.4, there exists a nonsingular (3×3)-matrix M over $\mathbb{F}$ such that $\det(M) = \frac{\det(A)}{\det(A')}$ and $\det(M) \cdot A' = MAM^T$. We have $\frac{A'}{\det(A')} = M \cdot \frac{A}{\det(A)} \cdot M^T$. So, the matrices $\frac{A}{\det(A)}$ and $\frac{A'}{\det(A')}$ are congruent.

Conversely, suppose that the matrices $\frac{A'}{\det(A')}$ and $\frac{A}{\det(A)}$ are congruent. Then there exists a nonsingular (3×3)-matrix M over $\mathbb{F}$ such that $\frac{A'}{\det(A')} = M \cdot \frac{A}{\det(A)} \cdot M^T$. This implies that $\frac{\det(A')}{\det(A')^3} = \det(M) \cdot \frac{\det(A)}{\det(A)^3} \cdot \det(M)$, i.e. $\det(M) \in \{\frac{\det(A)}{\det(A')}, -\frac{\det(A)}{\det(A')}\}$. So, $\det(M) \cdot A' = MAM^T$ or $\det(-M) \cdot A' = (-M)A(-M)^T$. Lemma 5.4 then implies that γ_B and $\gamma_{B'}$ are $Sp(V, f)$-equivalent. ∎

Lemma 5.6 *Let $A = (a_{ij})$, $A' = (a'_{ij})$ and $M = (m_{ij})$ be three nonsingular (3×3)-matrices over $\mathbb{F}$ such that $\det(M) \cdot A' = MAM^T$. Then $[a_{23} - a_{32}, a_{31} - a_{13}, a_{12} - a_{21}] = [a'_{23} - a'_{32}, a'_{31} - a'_{13}, a'_{12} - a'_{21}] \cdot M$.*

Proof. We follow the convention that subindices are taken modulo 3. For every $i \in \{1, 2, 3\}$, we have

$$\begin{aligned} a'_{i,i+1} &= \frac{1}{\det(M)} \cdot \sum_{1 \leq k,l \leq 3} m_{ik} a_{kl} m_{i+1,l}, \\ a'_{i+1,i} &= \frac{1}{\det(M)} \cdot \sum_{1 \leq k,l \leq 3} m_{i+1,k} a_{kl} m_{il}, \end{aligned}$$

and hence

$$a'_{i,i+1} - a'_{i+1,i} = \frac{1}{\det(M)} \cdot \sum_{1 \leq k,l \leq 3} (m_{ik} m_{i+1,l} - m_{i+1,k} m_{il}) a_{kl}.$$

For every $j \in \{1, 2, 3\}$, we have

$$\begin{aligned} & \sum_{i=1}^{3} (a'_{i,i+1} - a'_{i+1,i}) m_{i+2,j} \\ = \ & \frac{1}{\det(M)} \cdot \sum_{1 \leq k,l \leq 3} \Big(\sum_{i=1}^{3} (m_{ik} m_{i+1,l} m_{i+2,j} - m_{i+1,k} m_{il} m_{i+2,j}) \Big) a_{kl} \\ = \ & \frac{1}{\det(M)} \cdot \sum_{1 \leq k,l \leq 3} \mu_{klj} \cdot \det(M) \cdot a_{kl} \\ = \ & a_{j+1,j+2} - a_{j+2,j+1}. \end{aligned}$$

Here, $\mu_{klj} = 1$ if $(k, l, j) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, $\mu_{klj} = -1$ if $(k, l, j) \in \{(1, 3, 2), (2, 1, 3), (3, 2, 1)\}$ and $\mu_{klj} = 0$ otherwise. So, $[a_{23} - a_{32}, a_{31} - a_{13}, a_{12} - a_{21}] = [a'_{23} - a'_{32}, a'_{31} - a'_{13}, a'_{12} - a'_{21}] \cdot M$ as we needed to prove. ■

Lemma 5.7 *If $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ is an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is totally isotropic, then there exists an ordered basis $B' = (\bar{v}'_1, \bar{v}'_2, \bar{v}'_3, \bar{w}'_1, \bar{w}'_2, \bar{w}'_3)$ of V such that $\gamma_B = \gamma_{B'}$ and $f(\bar{v}'_1, \bar{w}'_1) \neq 0$.*

Proof. Suppose such an ordered basis B' of V does not exist. Then $f(\bar{v}_1, \bar{w}_1) = 0$. We also have $f(\bar{v}_2, \bar{w}_2) = 0$, otherwise the ordered basis $(\bar{v}_2, \bar{v}_3, \bar{v}_1, \bar{w}_2, \bar{w}_3, \bar{w}_1)$ would satisfy the required condition. We also have $f(\bar{v}_3, \bar{w}_3) = 0$, otherwise the ordered basis $(\bar{v}_3, \bar{v}_1, \bar{v}_2, \bar{w}_3, \bar{w}_1, \bar{w}_2)$ would satisfy the required condition.

Since $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = (\bar{v}_1 + \lambda \bar{v}_2 + \mu \bar{v}_3) \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge (\bar{w}_1 + \lambda \bar{w}_2 + \mu \bar{w}_3) + \bar{v}_3 \wedge (\bar{v}_1 + \lambda \bar{v}_2 + \mu \bar{v}_3) \wedge \bar{w}_2$, we have $0 = f(\bar{v}_1 + \lambda \bar{v}_2 + \mu \bar{v}_3, \bar{w}_1 + \lambda \bar{w}_2 + \mu \bar{w}_3) = f(\bar{v}_1, \bar{w}_1) + \lambda \cdot (f(\bar{v}_1, \bar{w}_2) + f(\bar{v}_2, \bar{w}_1)) + \mu \cdot (f(\bar{v}_3, \bar{w}_1) + f(\bar{v}_1, \bar{w}_3)) + \lambda\mu \cdot (f(\bar{v}_2, \bar{w}_3) +$

$f(\bar{v}_3, \bar{w}_2)) + \lambda^2 \cdot f(\bar{v}_2, \bar{w}_2) + \mu^2 \cdot f(\bar{v}_3, \bar{w}_3) = \lambda \cdot (f(\bar{v}_1, \bar{w}_2) + f(\bar{v}_2, \bar{w}_1)) + \mu \cdot (f(\bar{v}_3, \bar{w}_1) + f(\bar{v}_1, \bar{w}_3)) + \lambda\mu \cdot (f(\bar{v}_2, \bar{w}_3) + f(\bar{v}_3, \bar{w}_2))$ for all $\lambda, \mu \in \mathbb{F}$. Taking $\mu = 0$ and $\lambda \neq 0$, we find $f(\bar{v}_1, \bar{w}_2) + f(\bar{v}_2, \bar{w}_1) = 0$. Taking $\mu \neq 0$ and $\lambda = 0$, we find $f(\bar{v}_3, \bar{w}_1) + f(\bar{v}_1, \bar{w}_3) = 0$. So, we must also have that $f(\bar{v}_2, \bar{w}_3) + f(\bar{v}_3, \bar{w}_2) = 0$.

Suppose $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_2, \bar{w}_1) = 0$. Then $\langle \bar{w}_1, \bar{w}_2 \rangle \subseteq \langle \bar{v}_1, \bar{v}_2 \rangle^{\perp_f}$. Now, $\langle \bar{v}_1, \bar{v}_2 \rangle^{\perp_f}$ is a 4-dimensional subspace of V containing the 3-dimensional subspace $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ and the 2-dimensional subspace $\langle \bar{w}_1, \bar{w}_2 \rangle$. This is however impossible since $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{w}_1$ and $\bar{w}_2$ are linearly independent. Hence, $f(\bar{v}_1, \bar{w}_2) = -f(\bar{v}_2, \bar{w}_1) \neq 0$.

Since $f(\bar{v}_1, \bar{w}_2) \neq 0$, there exists a unique $\lambda \in \mathbb{F}$ such that $f(\bar{v}_3 + \lambda\bar{v}_1, \bar{w}_2) = f(\bar{v}_3, \bar{w}_2) + \lambda \cdot f(\bar{v}_1, \bar{w}_2) = 0$. For this value of λ, we also have $f(\bar{v}_2, \bar{w}_2) = 0$, $f(\bar{v}_3 + \lambda\bar{v}_1, \bar{w}_3 + \lambda\bar{w}_1) = f(\bar{v}_3, \bar{w}_3) + \lambda \cdot (f(\bar{v}_3, \bar{w}_1) + f(\bar{v}_1, \bar{w}_3)) + \lambda^2 \cdot f(\bar{v}_1, \bar{w}_1) = 0$ and $f(\bar{v}_2, \bar{w}_3 + \lambda\bar{w}_1) = f(\bar{v}_2, \bar{w}_3) + \lambda \cdot f(\bar{v}_2, \bar{w}_1) = -f(\bar{v}_3, \bar{w}_2) - \lambda \cdot f(\bar{v}_1, \bar{w}_2) = 0$. So, $\langle \bar{v}_2, \bar{v}_3 + \lambda\bar{v}_1 \rangle^{\perp_f}$ is a 4-dimensional subspace of V containing the 3-dimensional subspace $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ and the 2-dimensional subspace $\langle \bar{w}_2, \bar{w}_3 + \lambda\bar{w}_1 \rangle$. This is again impossible since $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, $\bar{w}_2$ and $\bar{w}_3 + \lambda\bar{w}_1$ are linearly independent. ∎

Remark. In Lemma 5.7, the fact that $\gamma_B = \gamma_{B'}$ implies that also the subspace $\langle \bar{v}'_1, \bar{v}'_2, \bar{v}'_3 \rangle$ is totally isotropic since it is equal to $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$. (Recall Lemma 3.1.) A similar remark applies to several of the lemmas below.

Lemma 5.8 *If $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ is an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is totally isotropic and $f(\bar{v}_1, \bar{w}_1) \neq 0$, then we can find an ordered basis $B' = (\bar{v}'_1, \bar{v}'_2, \bar{v}'_3, \bar{w}'_1, \bar{w}'_2, \bar{w}'_3)$ of V such that $\gamma_B = \gamma_{B'}$, $f(\bar{v}'_1, \bar{w}'_1) \neq 0$ and $f(\bar{v}'_1, \bar{w}'_2) = f(\bar{v}'_1, \bar{w}'_3) = 0$.*

Proof. We can take $\bar{v}'_1 := \bar{v}_1$, $\bar{w}'_1 = \bar{w}_1$ and

$$\bar{v}'_2 := \bar{v}_2 + \lambda\bar{v}_1, \quad \bar{v}'_3 := \bar{v}_3 + \mu\bar{v}_1, \quad \bar{w}'_2 := \bar{w}_2 + \lambda\bar{w}_1, \quad \bar{w}'_3 := \bar{w}_3 + \mu\bar{w}_1,$$

where

$$\lambda = -\frac{f(\bar{v}_1, \bar{w}_2)}{f(\bar{v}_1, \bar{w}_1)}, \qquad \mu = -\frac{f(\bar{v}_1, \bar{w}_3)}{f(\bar{v}_1, \bar{w}_1)}.$$

∎

Lemma 5.9 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is totally isotropic, $f(\bar{v}_1, \bar{w}_1) \neq 0$ and $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_1, \bar{w}_3) = f(\bar{v}_2, \bar{w}_2) = 0$. Then there exists an ordered basis $B' = (\bar{v}'_1, \bar{v}'_2, \bar{v}'_3, \bar{w}'_1, \bar{w}'_2, \bar{w}'_3)$ of V such that*

- *$\gamma_B = \gamma_{B'}$;*
- *$f(\bar{v}'_i, \bar{w}'_j) = f(\bar{v}_i, \bar{w}_j)$ for all $i, j \in \{1, 2, 3\}$;*
- *$\langle \bar{w}'_1, \bar{w}'_2, \bar{w}'_3 \rangle$ is totally isotropic.*

Proof. If $f(\bar{v}_3, \bar{w}_2) = 0$, then $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_2 \rangle$ would be a totally isotropic 4-space which is impossible. Hence, $f(\bar{v}_3, \bar{w}_2) \neq 0$. The lemma holds if we take $\bar{v}'_1 := \bar{v}_1$, $\bar{v}'_2 := \bar{v}_2$, $\bar{v}'_3 := \bar{v}_3$, $\bar{w}'_2 := \bar{w}_2$ and

$$\bar{w}'_1 := \bar{w}_1 + b\bar{v}_1 + c\bar{v}_3, \qquad \bar{w}'_3 := \bar{w}_3 + a\bar{v}_1 - b\bar{v}_3,$$

where

$$b = \frac{f(\bar{w}_2, \bar{w}_3)}{f(\bar{w}_2, \bar{v}_3)}, \qquad c = -\frac{f(\bar{w}_2, \bar{w}_1)}{f(\bar{w}_2, \bar{v}_3)},$$

$$a = \frac{b \cdot f(\bar{w}_1, \bar{v}_3) - c \cdot f(\bar{v}_3, \bar{w}_3) - f(\bar{w}_1, \bar{w}_3)}{f(\bar{w}_1, \bar{v}_1)}.$$

■

Lemma 5.10 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is totally isotropic, $f(\bar{v}_1, \bar{w}_1) \neq 0$, $f(\bar{v}_2, \bar{w}_2) \neq 0$ and $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_1, \bar{w}_3) = 0$. Then we can find an ordered basis $B' = (\bar{v}'_1, \bar{v}'_2, \bar{v}'_3, \bar{w}'_1, \bar{w}'_2, \bar{w}'_3)$ of V such that $\gamma_B = \gamma_{B'}$, $f(\bar{v}'_1, \bar{w}'_1) \neq 0$, $f(\bar{v}'_2, \bar{w}'_2) \neq 0$, $f(\bar{v}'_1, \bar{w}'_2) = f(\bar{v}'_1, \bar{w}'_3) = f(\bar{v}'_2, \bar{w}'_3) = 0$.*

Proof. We can take $\bar{v}'_1 := \bar{v}_1$, $\bar{v}'_2 := \bar{v}_2$, $\bar{w}'_1 := \bar{w}_1$, $\bar{w}'_2 := \bar{w}_2$, $\bar{v}'_3 := \bar{v}_3 + \lambda \bar{v}_2$ and $\bar{w}'_3 := \bar{w}_3 + \lambda \bar{w}_2$, where $\lambda = -\frac{f(\bar{v}_2, \bar{w}_3)}{f(\bar{v}_2, \bar{w}_2)}$. ■

Lemma 5.11 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is totally isotropic, $f(\bar{v}_1, \bar{w}_1) \neq 0$, $f(\bar{v}_2, \bar{w}_2) \neq 0$ and $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_1, \bar{w}_3) = f(\bar{v}_2, \bar{w}_3) = 0$. Then there exists an ordered basis $B' = (\bar{v}'_1, \bar{v}'_2, \bar{v}'_3, \bar{w}'_1, \bar{w}'_2, \bar{w}'_3)$ of V such that*

- *$\gamma_B = \gamma_{B'}$;*
- *$f(\bar{v}'_1, \bar{w}'_1) \neq 0$, $f(\bar{v}'_2, \bar{w}'_2) \neq 0$ and $f(\bar{v}'_1, \bar{w}'_2) = f(\bar{v}'_1, \bar{w}'_3) = f(\bar{v}'_2, \bar{w}'_3) = 0$;*
- *$\langle \bar{w}'_1, \bar{w}'_2, \bar{w}'_3 \rangle$ is totally isotropic.*

Proof. If $f(\bar{v}_3, \bar{w}_3) = 0$, then $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_3 \rangle$ would be a totally isotropic 4-space which is impossible. Hence, $f(\bar{v}_3, \bar{w}_3) \neq 0$. The lemma holds if we take $\bar{v}'_1 := \bar{v}_1$, $\bar{v}'_2 := \bar{v}_2$, $\bar{v}'_3 := \bar{v}_3$ and

$$\bar{w}'_1 := \bar{w}_1 + a\bar{v}_2, \qquad \bar{w}'_2 := \bar{w}_2 + b\bar{v}_3, \qquad \bar{w}'_3 := \bar{w}_3 + c\bar{v}_1,$$

where

$$c = -\frac{f(\bar{w}_1, \bar{w}_3)}{f(\bar{w}_1, \bar{v}_1)}, \qquad b = -\frac{f(\bar{w}_2, \bar{w}_3)}{f(\bar{v}_3, \bar{w}_3)}, \qquad a = -\frac{f(\bar{w}_1, \bar{w}_2) + b \cdot f(\bar{w}_1, \bar{v}_3)}{f(\bar{v}_2, \bar{w}_2)}.$$

■

Lemma 5.12 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is totally isotropic, $f(\bar{v}_1, \bar{w}_1) \neq 0$ and $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_1, \bar{w}_3) = 0$. Suppose there exists no ordered basis $B' = (\bar{v}'_1, \bar{v}'_2, \bar{v}'_3, \bar{w}'_1, \bar{w}'_2, \bar{w}'_3)$ of V such that $\gamma_{B'} = \gamma_B$, $f(\bar{v}'_1, \bar{w}'_1) \neq 0$, $f(\bar{v}'_1, \bar{w}'_2) = f(\bar{v}'_1, \bar{w}'_3) = 0$ and $f(\bar{v}'_2, \bar{w}'_2) \neq 0$. Then the following must hold.*

(1) $f(\bar{v}_2, \bar{w}_2) = f(\bar{v}_3, \bar{w}_3) = f(\bar{v}_2, \bar{w}_3) + f(\bar{v}_3, \bar{w}_2) = 0$.

(2) *There exists an ordered basis $B'' = (\bar{v}''_1, \bar{v}''_2, \bar{v}''_3, \bar{w}''_1, \bar{w}''_2, \bar{w}''_3)$ of V such that $\gamma_{B''} = \gamma_B$, $f(\bar{v}''_1, \bar{w}''_1) \neq 0$, $f(\bar{v}''_1, \bar{w}''_2) = f(\bar{v}''_1, \bar{w}''_3) = f(\bar{v}''_2, \bar{w}''_2) = f(\bar{v}''_3, \bar{w}''_3) = f(\bar{v}''_2, \bar{w}''_3) + f(\bar{v}''_3, \bar{w}''_2) = f(\bar{v}''_2, \bar{w}''_1) = 0$ and $\langle \bar{w}''_1, \bar{w}''_2, \bar{w}''_3 \rangle$ is totally isotropic.*

Proof. Let α and β be arbitrary elements of $\mathbb{F}$ such that $(\alpha,\beta) \neq (0,0)$. Then there exist $\gamma,\delta \in \mathbb{F}$ such that $\alpha\delta-\beta\gamma = 1$. If we put $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3') = (\bar{v}_1, \alpha\bar{v}_2+\beta\bar{v}_3, \gamma\bar{v}_2+\delta\bar{v}_3, \bar{w}_1, \alpha\bar{w}_2+\beta\bar{w}_3, \gamma\bar{w}_2+\delta\bar{w}_3)$, then $\gamma_{B'} = \gamma_B$, $f(\bar{v}_1', \bar{w}_1') \neq 0$ and $f(\bar{v}_1', \bar{w}_2') = f(\bar{v}_1', \bar{w}_3') = 0$. So, we must have that $f(\bar{v}_2', \bar{w}_2') = \alpha^2 f(\bar{v}_2, \bar{w}_2)+\alpha\beta(f(\bar{v}_2, \bar{w}_3)+f(\bar{v}_3, \bar{w}_2))+\beta^2 f(\bar{v}_3, \bar{w}_3) = 0$ for all $\alpha,\beta \in \mathbb{F}$ such that $(\alpha,\beta) \neq (0,0)$. This implies that $f(\bar{v}_2, \bar{w}_2) = f(\bar{v}_3, \bar{w}_3) = f(\bar{v}_2, \bar{w}_3) + f(\bar{v}_3, \bar{w}_2) = 0$.

Now, choose $\alpha,\beta \in \mathbb{F}$ such that $(\alpha,\beta) \neq (0,0)$ and $f(\alpha\bar{v}_2 + \beta\bar{v}_3, \bar{w}_1) = 0$, and let $\gamma,\delta \in \mathbb{F}$ such that $\alpha\delta - \beta\gamma = 1$. Then the ordered basis $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3') = (\bar{v}_1, \alpha\bar{v}_2 + \beta\bar{v}_3, \gamma\bar{v}_2 + \delta\bar{v}_3, \bar{w}_1, \alpha\bar{w}_2 + \beta\bar{w}_3, \gamma\bar{w}_2 + \delta\bar{w}_3)$ of V satisfies all required conditions except maybe for the fact that $\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle$ is totally isotropic. However, by Lemma 5.9 we can construct an ordered basis of V for which this final condition is also satisfied. ■

Corollary 5.13 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is totally isotropic. Then there exists an ordered basis $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ of V such that*

- $\gamma_B = \gamma_{B'}$;
- $f(\bar{v}_1', \bar{w}_1') \neq 0$ *and* $f(\bar{v}_1', \bar{w}_2') = f(\bar{v}_1', \bar{w}_3') = 0$;
- $\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle$ *is totally isotropic;*
- *either* $(f(\bar{v}_2', \bar{w}_3') = 0$ *and* $f(\bar{v}_2', \bar{w}_2') \neq 0)$ *or* $(f(\bar{v}_2', \bar{w}_2') = f(\bar{v}_3', \bar{w}_3') = f(\bar{v}_2', \bar{w}_3') + f(\bar{v}_3', \bar{w}_2') = f(\bar{v}_2', \bar{w}_1') = 0)$.

Proof. By Lemmas 5.7 and 5.8, we may assume that $f(\bar{v}_1, \bar{w}_1) \neq 0$ and $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_1, \bar{w}_3) = 0$. If there exists an ordered basis $B'' = (\bar{v}_1'', \bar{v}_2'', \bar{v}_3'', \bar{w}_1'', \bar{w}_2'', \bar{w}_3'')$ of V such that $\gamma_{B''} = \gamma_B$, $f(\bar{v}_1'', \bar{w}_1'') \neq 0$, $f(\bar{v}_1'', \bar{w}_2'') = f(\bar{v}_1'', \bar{w}_3'') = 0$ and $f(\bar{v}_2'', \bar{w}_2'') \neq 0$, then Lemmas 5.10 and 5.11 imply the existence of the ordered basis B' of V having the required properties. If there exists no such ordered basis B'', then the existence of the required ordered basis B' is guaranteed by Lemma 5.12. ■

Lemma 5.14 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ and $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ are totally isotropic, $f(\bar{v}_1, \bar{w}_1) \neq 0 \neq f(\bar{v}_2, \bar{w}_2)$ and $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_1, \bar{w}_3) = f(\bar{v}_2, \bar{w}_3) = 0$. Then there exists an ordered basis $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ of V such that $\gamma_{B'} = \gamma_B$, $f(\bar{v}_1', \bar{w}_1') \neq 0 \neq f(\bar{v}_2', \bar{w}_2')$, $f(\bar{v}_1', \bar{w}_2') = f(\bar{v}_1', \bar{w}_3') = f(\bar{v}_2', \bar{w}_3') = 0$, $\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle$ is totally isotropic and at least one of the following four conditions is satisfied:*

(1) $f(\bar{v}_2', \bar{w}_1') = f(\bar{v}_3', \bar{w}_1') = f(\bar{v}_3', \bar{w}_2') = 0$;

(2) $f(\bar{v}_2', \bar{w}_1') = f(\bar{v}_3', \bar{w}_2') = 0$ *and* $f(\bar{v}_3', \bar{w}_1') \neq 0$;

(3) $f(\bar{v}_2', \bar{w}_1') = 0$ *and* $f(\bar{v}_3', \bar{w}_1') \neq 0 \neq f(\bar{v}_3', \bar{w}_2')$;

(4) $f(\bar{v}_2', \bar{w}_1') \neq 0 \neq f(\bar{v}_3', \bar{w}_1')$ *and* $f(\bar{v}_3', \bar{w}_1')^2 \cdot f(\bar{v}_2', \bar{w}_2') - f(\bar{v}_2', \bar{w}_1') \cdot f(\bar{v}_3', \bar{w}_1') \cdot f(\bar{v}_3', \bar{w}_2') + f(\bar{v}_2', \bar{w}_1')^2 \cdot f(\bar{v}_3', \bar{w}_3') = 0$.

Proof. Since $f(\bar{v}_1, \bar{w}_3) = f(\bar{v}_2, \bar{w}_3) = 0$ and $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_3 \rangle$ is not totally isotropic, we necessarily have $f(\bar{v}_3, \bar{w}_3) \neq 0$.

If $f(\bar{v}_2, \bar{w}_1) = 0$, then the ordered basis $B' = B$ satisfies the conditions of the lemma (with either case (1), (2) or (3) occurring), except in the case where $f(\bar{v}_2, \bar{w}_1) = f(\bar{v}_3, \bar{w}_1) = 0$ and $f(\bar{v}_3, \bar{w}_2) \neq 0$. In the latter case, the ordered basis $B' = (\bar{v}_2, \bar{v}_1, \bar{v}_3, -\bar{w}_2, -\bar{w}_1, -\bar{w}_3)$ satisfies the required properties (with case (2) occurring). So, in the sequel we may suppose that $f(\bar{v}_2, \bar{w}_1) \neq 0$.

Suppose $f(\bar{v}_3, \bar{w}_1)^2 \cdot f(\bar{v}_2, \bar{w}_2) - f(\bar{v}_2, \bar{w}_1) \cdot f(\bar{v}_3, \bar{w}_1) \cdot f(\bar{v}_3, \bar{w}_2) + f(\bar{v}_2, \bar{w}_1)^2 \cdot f(\bar{v}_3, \bar{w}_3) = 0$. Then the fact that $f(\bar{v}_2, \bar{w}_1)$ and $f(\bar{v}_3, \bar{w}_3)$ are distinct from 0 implies that $f(\bar{v}_3, \bar{w}_1) \neq 0$. So, the ordered basis $B' = B$ satisfies the conditions of the lemma with case (4) occurring. So, in the sequel, we may also suppose that $f(\bar{v}_3, \bar{w}_1)^2 \cdot f(\bar{v}_2, \bar{w}_2) - f(\bar{v}_2, \bar{w}_1) \cdot f(\bar{v}_3, \bar{w}_1) \cdot f(\bar{v}_3, \bar{w}_2) + f(\bar{v}_2, \bar{w}_1)^2 \cdot f(\bar{v}_3, \bar{w}_3) \neq 0$.

Put $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3') = (\bar{v}_1, \alpha\bar{v}_2 + \beta\bar{v}_3, \gamma\bar{v}_2 + \delta\bar{v}_3, \frac{\bar{w}_1}{\Delta}, \frac{\alpha\bar{w}_2 + \beta\bar{w}_3}{\Delta}, \frac{\gamma\bar{w}_2 + \delta\bar{w}_3}{\Delta})$, where α, β, γ and δ are elements of $\mathbb{F}$, to be chosen later, such that $\Delta := \alpha\delta - \beta\gamma \neq 0$. Clearly, $\gamma_{B'} = \gamma_B$, $f(\bar{v}_1', \bar{w}_1') \neq 0$, $f(\bar{v}_1', \bar{w}_2') = f(\bar{v}_1', \bar{w}_3') = 0$ and $\langle \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle$ is totally isotropic.

We now show that we can choose α, β, γ and δ in such a way that $f(\bar{v}_2', \bar{w}_1') = f(\bar{v}_2', \bar{w}_3') = 0$. Since $\langle \bar{v}_2', \bar{w}_1', \bar{w}_2', \bar{w}_3' \rangle$ is not totally isotropic, this also implies that $f(\bar{v}_2', \bar{w}_2') \neq 0$. The fact that $f(\bar{v}_2', \bar{w}_1') = 0$ implies by the above discussion that the conditions of the lemma hold for the ordered basis $B' = (\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{w}_1', \bar{w}_2', \bar{w}_3')$ or the ordered basis $(\bar{v}_2', \bar{v}_1', \bar{v}_3', -\bar{w}_2', -\bar{w}_1', -\bar{w}_3')$.

The condition $f(\bar{v}_2', \bar{w}_1') = 0$ requires that

$$\alpha = -\frac{f(\bar{v}_3, \bar{w}_1)}{f(\bar{v}_2, \bar{w}_1)} \cdot \beta, \tag{7}$$

with $\beta \neq 0$. Now, we have $\Delta \cdot f(\bar{v}_2', \bar{w}_3') = f(\alpha\bar{v}_2 + \beta\bar{v}_3, \gamma\bar{w}_2 + \delta\bar{w}_3) = \alpha\gamma \cdot f(\bar{v}_2, \bar{w}_2) + \beta\gamma \cdot f(\bar{v}_3, \bar{w}_2) + \beta\delta \cdot f(\bar{v}_3, \bar{w}_3)$. In view of equation (7), the condition $f(\bar{v}_2', \bar{w}_3') = 0$ requires that

$$\delta = \frac{1}{f(\bar{v}_3, \bar{w}_3)} \cdot \Big(\frac{f(\bar{v}_3, \bar{w}_1)}{f(\bar{v}_2, \bar{w}_1)} f(\bar{v}_2, \bar{w}_2) - f(\bar{v}_3, \bar{w}_2) \Big) \cdot \gamma, \tag{8}$$

with $\gamma \neq 0$. If we choose α, β, γ and δ as indicated in equations (7) and (8), then the condition that $f(\bar{v}_3, \bar{w}_1)^2 \cdot f(\bar{v}_2, \bar{w}_2) - f(\bar{v}_2, \bar{w}_1) \cdot f(\bar{v}_3, \bar{w}_1) \cdot f(\bar{v}_3, \bar{w}_2) + f(\bar{v}_2, \bar{w}_1)^2 \cdot f(\bar{v}_3, \bar{w}_3) \neq 0$ implies that $\Delta = \alpha\delta - \beta\gamma \neq 0$. So, such α, β, γ and δ satisfy all required conditions. ∎

Lemma 5.15 *Let $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ be an ordered basis of V such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ and $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ are totally isotropic with respect to f. Let $\mu_1, \mu_2, \mu_3 \in \mathbb{F}^*$. Then the trivector $\mu_3 \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \mu_1 \cdot \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \mu_2 \cdot \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2$ is $Sp(V, f)$-equivalent with $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \frac{\mu_1}{\mu_3} \cdot \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \frac{\mu_2}{\mu_3} \cdot \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2$.*

Proof. Since $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ and $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ are totally isotropic subspaces of V, we can take a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) such that $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle = \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$ and $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{f}_1, \bar{f}_2, \bar{f}_3 \rangle$. Let θ be the element of $Sp(V, f)$ which maps the hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) to the hyperbolic basis $(\frac{\bar{e}_1}{\mu_3}, \mu_3\bar{f}_1, \frac{\bar{e}_2}{\mu_3}, \mu_3\bar{f}_2, \frac{\bar{e}_3}{\mu_3}, \mu_3\bar{f}_3)$ of (V, f). Then $\theta(\bar{v}) = \frac{\bar{v}}{\mu_3}$ and $\theta(\bar{w}) = \mu_3\bar{w}$ for every $\bar{v} \in \langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ and every $\bar{w} \in \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$. Clearly, we have $\bigwedge^3(\theta)(\mu_3 \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \mu_1 \cdot \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \mu_2 \cdot \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2) = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \frac{\mu_1}{\mu_3} \cdot \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \frac{\mu_2}{\mu_3} \cdot \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2$. ∎

As in Section 1, let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a fixed hyperbolic basis of (V, f).

Lemma 5.16 *Let χ be a trivector of Type (D) whose base 3-space is totally isotropic. Then χ is $Sp(V,f)$-equivalent with (at least) one of the following trivectors:*

- $\alpha_1(\lambda_1, \lambda_2) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$*;*
- $\alpha_2(\lambda_1, \lambda_2) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$*;*
- $\alpha_3(\lambda_1, \lambda_2) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_2^* + \bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge ((\lambda_2 - 1)\bar{f}_2^* + \lambda_2 \bar{f}_3^*)$ *for some* $\lambda_1 \in \mathbb{F}^*$ *and some* $\lambda_2 \in \mathbb{F} \setminus \{1\}$*;*
- $\alpha_4(\lambda_1, \lambda_2) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge (\bar{f}_2^* + \bar{f}_3^*)$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$*;*
- $\beta_1 = -\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$*;*
- $\beta_2 = -\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$.

Proof. By Corollary 5.13, χ can be written in the form $\gamma_B = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2$ where $B = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3)$ is an ordered basis of V for which the following holds:

- $f(\bar{v}_1, \bar{w}_1) \neq 0$ and $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_1, \bar{w}_3) = 0$,
- $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ and $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ are totally isotropic,
- either $(f(\bar{v}_2, \bar{w}_3) = 0$ and $f(\bar{v}_2, \bar{w}_2) \neq 0)$ or $(f(\bar{v}_2, \bar{w}_2) = f(\bar{v}_3, \bar{w}_3) = f(\bar{v}_2, \bar{w}_3) + f(\bar{v}_3, \bar{w}_2) = f(\bar{v}_2, \bar{w}_1) = 0)$.

If $f(\bar{v}_2, \bar{w}_3) = 0$ and $f(\bar{v}_2, \bar{w}_2) \neq 0$, then by Lemma 5.14, we may moreover assume that one of the cases (1)–(4) of that lemma are satisfied for the ordered basis B.

Suppose $\{i,j,k\} = \{1,2,3\}$. If $\bar{w} \in \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \cap \bar{v}_1^{\perp_f} \cap \bar{v}_2^{\perp_f} \cap \bar{v}_3^{\perp_f}$, then $\langle \bar{w}, \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ would be a totally isotropic 4-space which is impossible. Hence, $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle \cap \bar{v}_1^{\perp_f} \cap \bar{v}_2^{\perp_f} \cap \bar{v}_3^{\perp_f} = \{\bar{o}\}$. This implies that $\bar{v}_i^{\perp_f} \cap \bar{v}_j^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ is a one-dimensional subspace $\langle \bar{u}_k \rangle$ of $\langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$. The vector $\bar{u}_k$ is uniquely determined, up to a nonzero factor of $\mathbb{F}$. Since $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle \subseteq \bar{v}_i^{\perp_f} \neq V$, the subspace $\bar{v}_i^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ must be equal to the two-dimensional subspace $\langle \bar{u}_j, \bar{u}_k \rangle$. We have $\langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle = \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle$ and $f(\bar{v}_i, \bar{u}_i) \neq 0$ for every $i \in \{1,2,3\}$.

(1) Suppose first that $f(\bar{v}_2, \bar{w}_3) = 0$ and $f(\bar{v}_2, \bar{w}_2) \neq 0$. Then from $f(\bar{v}_1, \bar{w}_2) = f(\bar{v}_1, \bar{w}_3) = f(\bar{v}_2, \bar{w}_3) = 0$, we immediately see that $\langle \bar{u}_3 \rangle = \bar{v}_1^{\perp_f} \cap \bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{w}_3 \rangle$ and $\langle \bar{u}_2, \bar{u}_3 \rangle = \bar{v}_1^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{w}_2, \bar{w}_3 \rangle$. So, the vectors $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$ can be taken in such a way that

$$\bar{w}_3 = \bar{u}_3, \qquad \bar{w}_2 = \bar{u}_2 + a\bar{u}_3, \qquad \bar{w}_1 = \bar{u}_1 + b\bar{u}_2 + c\bar{u}_3$$

for some $a, b, c \in \mathbb{F}$. We have $f(\bar{v}_3, \bar{w}_2) = a \cdot f(\bar{v}_3, \bar{u}_3)$, $f(\bar{v}_2, \bar{w}_1) = b \cdot f(\bar{v}_2, \bar{u}_2)$ and $f(\bar{v}_3, \bar{w}_1) = c \cdot f(\bar{v}_3, \bar{u}_3)$. We now consider the four cases corresponding to the four cases of Lemma 5.14.

- Suppose $f(\bar{v}_2, \bar{w}_1) = f(\bar{v}_3, \bar{w}_1) = f(\bar{v}_3, \bar{w}_2) = 0$. Then $(a,b,c) = (0,0,0)$ and we define $(\bar{f}_1', \bar{f}_2', \bar{f}_3') := (\bar{u}_1, \bar{u}_2, \bar{u}_3)$.
- Suppose $f(\bar{v}_2, \bar{w}_1) = f(\bar{v}_3, \bar{w}_2) = 0$ and $f(\bar{v}_3, \bar{w}_1) \neq 0$. Then $(a,b) = (0,0)$ and $c \neq 0$. We define $(\bar{f}_1', \bar{f}_2', \bar{f}_3') := (\bar{u}_1, \bar{u}_2, c\bar{u}_3)$.

• Suppose $f(\bar{v}_2, \bar{w}_1) = 0$ and $f(\bar{v}_3, \bar{w}_1) \neq 0 \neq f(\bar{v}_3, \bar{w}_2)$. Then $b = 0$ and $a \neq 0 \neq c$. We define $(\bar{f}_1', \bar{f}_2', \bar{f}_3') := (a\bar{u}_1, c\bar{u}_2, ac\bar{u}_3)$.

• Suppose $f(\bar{v}_2, \bar{w}_1) \neq 0 \neq f(\bar{v}_3, \bar{w}_1)$ and $f(\bar{v}_3, \bar{w}_1)^2 \cdot f(\bar{v}_2, \bar{w}_2) - f(\bar{v}_2, \bar{w}_1) \cdot f(\bar{v}_3, \bar{w}_1) \cdot f(\bar{v}_3, \bar{w}_2) + f(\bar{v}_2, \bar{w}_1)^2 \cdot f(\bar{v}_3, \bar{w}_3) = 0$. Then $b \neq 0 \neq c$. We define $(\bar{f}_1', \bar{f}_2', \bar{f}_3') := (\bar{u}_1, b\bar{u}_2, c\bar{u}_3)$.

Since $f(\bar{v}_1, \bar{u}_1)$, $f(\bar{v}_2, \bar{u}_2)$ and $f(\bar{v}_3, \bar{u}_3)$ are distinct from 0, there exist unique vectors $\bar{e}_1' \in \langle \bar{v}_1 \rangle$, $\bar{e}_2' \in \langle \bar{v}_2 \rangle$ and $\bar{e}_3' \in \langle \bar{v}_3 \rangle$ such that $f(\bar{e}_1', \bar{f}_1') = f(\bar{e}_2', \bar{f}_2') = f(\bar{e}_3', \bar{f}_3') = 1$. Then $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ is a hyperbolic basis of (V, f).

• Suppose $(a, b, c) = (0, 0, 0)$. Then there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \mu_1 \cdot \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \mu_2 \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge \bar{f}_1' + \mu_3 \cdot \bar{e}_3' \wedge \bar{e}_1' \wedge \bar{f}_2'$. Lemma 5.15 then implies that χ is $Sp(V, f)$-equivalent with $\bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \frac{\mu_2}{\mu_1} \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge \bar{f}_1' + \frac{\mu_3}{\mu_1} \cdot \bar{e}_3' \wedge \bar{e}_1' \wedge \bar{f}_2'$, i.e. $Sp(V, f)$-equivalent with the trivector $\alpha_1(\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_1})$.

• Suppose $(a, b) = (0, 0)$ and $c \neq 0$. Then there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \mu_1 \cdot \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \mu_2 \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge (\bar{f}_1' + \bar{f}_3') + \mu_3 \cdot \bar{e}_3' \wedge \bar{e}_1' \wedge \bar{f}_2'$. Lemma 5.15 then implies that χ is $Sp(V, f)$-equivalent with $\bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \frac{\mu_2}{\mu_1} \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge (\bar{f}_1' + \bar{f}_3') + \frac{\mu_3}{\mu_1} \cdot \bar{e}_3' \wedge \bar{e}_1' \wedge \bar{f}_2'$, i.e. $Sp(V, f)$-equivalent with the trivector $\alpha_2(\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_1})$.

• Suppose $b = 0$ and $a \neq 0 \neq c$. Then there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \mu_1 \cdot \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \mu_2 \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge (\bar{f}_1' + \bar{f}_3') + \mu_3 \cdot \bar{e}_3' \wedge \bar{e}_1' \wedge (\bar{f}_2' + \bar{f}_3')$. Lemma 5.15 then implies that χ is $Sp(V, f)$-equivalent with $\bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \frac{\mu_2}{\mu_1} \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge (\bar{f}_1' + \bar{f}_3') + \frac{\mu_3}{\mu_1} \cdot \bar{e}_3' \wedge \bar{e}_1' \wedge (\bar{f}_2' + \bar{f}_3')$, i.e. $Sp(V, f)$-equivalent with the trivector $\alpha_4(\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_1})$.

• Suppose b and c are distinct from 0. Then there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{F}^*$ and a $\lambda \in \mathbb{F}$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \mu_1 \cdot \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \mu_2 \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge (\bar{f}_1' + \bar{f}_2' + \bar{f}_3') + \mu_3 \cdot \bar{e}_3' \wedge \bar{e}_1' \wedge (\bar{f}_2' + \lambda \bar{f}_3')$. From $f(\bar{v}_3, \bar{w}_1)^2 f(\bar{v}_2, \bar{w}_2) - f(\bar{v}_2, \bar{w}_1) f(\bar{v}_3, \bar{w}_1) f(\bar{v}_3, \bar{w}_2) + f(\bar{v}_2, \bar{w}_1)^2 f(\bar{v}_3, \bar{w}_3) = 0$ we have that $\frac{\mu_3}{\mu_1}\lambda = \frac{\mu_3}{\mu_1} + 1$. Lemma 5.15 then implies that χ is $Sp(V, f)$-equivalent with $\bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \frac{\mu_2}{\mu_1} \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge (\bar{f}_1' + \bar{f}_2' + \bar{f}_3') + \bar{e}_3' \wedge \bar{e}_1' \wedge (\frac{\mu_3}{\mu_1} \bar{f}_2' + (\frac{\mu_3}{\mu_1} + 1) \bar{f}_3')$, i.e. $Sp(V, f)$-equivalent with the trivector $\alpha_3(\frac{\mu_2}{\mu_1}, \frac{\mu_3}{\mu_1} + 1)$.

(2) Suppose $f(\bar{v}_2, \bar{w}_2) = f(\bar{v}_3, \bar{w}_3) = f(\bar{v}_2, \bar{w}_3) + f(\bar{v}_3, \bar{w}_2) = f(\bar{v}_2, \bar{w}_1) = 0$. We then immediately see that $\langle \bar{u}_3 \rangle = \bar{v}_1^{\perp_f} \cap \bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{w}_2 \rangle$, $\langle \bar{u}_2 \rangle = \bar{v}_1^{\perp_f} \cap \bar{v}_3^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{w}_3 \rangle$ and $\langle \bar{u}_1, \bar{u}_3 \rangle = \bar{v}_2^{\perp_f} \cap \langle \bar{w}_1, \bar{w}_2, \bar{w}_3 \rangle = \langle \bar{w}_1, \bar{w}_2 \rangle$. So, the vectors $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$ can be taken in such a way that

$$\bar{w}_3 = \bar{u}_2, \qquad \bar{w}_2 = \bar{u}_3, \qquad \bar{w}_1 = \bar{u}_1 + a\bar{u}_3$$

for some $a \in \mathbb{F}$.

• If $a = 0$, then we define $(\bar{f}_1', \bar{f}_2', \bar{f}_3') := (\bar{u}_1, \bar{u}_2, \bar{u}_3)$.

• If $a \neq 0$, then we define $(\bar{f}_1', \bar{f}_2', \bar{f}_3') := (\bar{u}_1, \bar{u}_2, a\bar{u}_3)$.

Since $f(\bar{v}_1, \bar{u}_1)$, $f(\bar{v}_2, \bar{u}_2)$ and $f(\bar{v}_3, \bar{u}_3)$ are distinct from 0, there exist unique vectors $\bar{e}_1' \in \langle \bar{v}_1 \rangle$, $\bar{e}_2' \in \langle \bar{v}_2 \rangle$ and $\bar{e}_3' \in \langle \bar{v}_3 \rangle$ such that $f(\bar{e}_1', \bar{f}_1') = f(\bar{e}_2', \bar{f}_2') = f(\bar{e}_3', \bar{f}_3') = 1$. Then $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ is a hyperbolic basis of (V, f).

• Suppose $a = 0$. Then there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \mu_1 \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 + \mu_2 \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge \bar{f}'_1 + \mu_3 \cdot \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_3$. From $f(\bar{v}_2, \bar{w}_3) + f(\bar{v}_3, \bar{w}_2) = 0$, we have $\mu_1 = -\mu_3$. Lemma 5.15 then implies that χ is $Sp(V, f)$-equivalent with $-\bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 + \frac{\mu_2}{\mu_3} \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge \bar{f}'_1 + \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_3$. Now, $-\bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 + \frac{\mu_2}{\mu_3} \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge \bar{f}'_1 + \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_3 = -\bar{e}''_1 \wedge \bar{e}''_2 \wedge \bar{f}''_2 + \bar{e}''_2 \wedge \bar{e}''_3 \wedge \bar{f}''_1 + \bar{e}''_3 \wedge \bar{e}''_1 \wedge \bar{f}''_3$, where $(\bar{e}''_1, \bar{f}''_1, \bar{e}''_2, \bar{f}''_2, \bar{e}''_3, \bar{f}''_3)$ is the hyperbolic basis $(\bar{e}'_1, \bar{f}'_1, \frac{\mu_2}{\mu_3}\bar{e}'_2, \frac{\mu_3}{\mu_2}\bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ of (V, f). So, χ is $Sp(V, f)$-equivalent with β_1.

• Suppose $a \neq 0$. Then there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_1 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{w}_2 = \mu_1 \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 + \mu_2 \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge (\bar{f}'_1 + \bar{f}'_3) + \mu_3 \cdot \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_3$. From $f(\bar{v}_2, \bar{w}_3) + f(\bar{v}_3, \bar{w}_2) = 0$, we have $\mu_1 = -\mu_3$. Lemma 5.15 then implies that χ is $Sp(V, f)$-equivalent with $-\bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 + \frac{\mu_2}{\mu_3} \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge (\bar{f}'_1 + \bar{f}'_3) + \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_3$. Now, $-\bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 + \frac{\mu_2}{\mu_3} \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge (\bar{f}'_1 + \bar{f}'_3) + \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_3 = -\bar{e}''_1 \wedge \bar{e}''_2 \wedge \bar{f}''_2 + \bar{e}''_2 \wedge \bar{e}''_3 \wedge (\bar{f}''_1 + \bar{f}''_3) + \bar{e}''_3 \wedge \bar{e}''_1 \wedge \bar{f}''_3$, where $(\bar{e}''_1, \bar{f}''_1, \bar{e}''_2, \bar{f}''_2, \bar{e}''_3, \bar{f}''_3)$ is the hyperbolic basis $(\bar{e}'_1, \bar{f}'_1, \frac{\mu_2}{\mu_3}\bar{e}'_2, \frac{\mu_3}{\mu_2}\bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ of (V, f). So, χ is $Sp(V, f)$-equivalent with β_2. ∎

In the following lemmas, we frequently make use of the polyvectors $\pi(\chi)$ and $\pi(\chi \wedge \pi(\chi))$, where χ is one of the trivectors mentioned in Lemma 5.16. We therefore list these trivectors below. If $\lambda_1, \lambda_2 \in \mathbb{F}^*$ and $\lambda'_2 \in \mathbb{F} \setminus \{0, 1\}$, then:

• $\pi(\alpha_1(\lambda_1, \lambda_2)) = 0$ and $\pi(\alpha_1(\lambda_1, \lambda_2) \wedge \pi(\alpha_1(\lambda_1, \lambda_2))) = 0$;

• $\pi(\alpha_2(\lambda_1, \lambda_2)) = \lambda_1 \bar{e}^*_2$ and $\pi(\alpha_2(\lambda_1, \lambda_2) \wedge \pi(\alpha_2(\lambda_1, \lambda_2))) = \lambda_1 \lambda_2 \cdot \bar{e}^*_1 \wedge \bar{e}^*_3$;

• $\pi(\alpha_3(\lambda_1, 0)) = \lambda_1 \bar{e}^*_2 - \lambda_1 \bar{e}^*_3$ and $\pi(\alpha_3(\lambda_1, 0) \wedge \pi(\alpha_3(\lambda_1, 0))) = \lambda_1 \cdot \bar{e}^*_1 \wedge (\bar{e}^*_2 - \bar{e}^*_3)$;

• $\pi(\alpha_3(\lambda_1, \lambda'_2)) = \lambda_1 \bar{e}^*_2 - \lambda_1 \bar{e}^*_3 - \lambda'_2 \bar{e}^*_1$ and $\pi(\alpha_3(\lambda_1, \lambda'_2) \wedge \pi(\alpha_3(\lambda_1, \lambda'_2))) = \frac{\lambda_1}{\lambda'_2} \cdot (\bar{e}^*_1 - \lambda'_2 \bar{e}^*_2) \wedge (\bar{e}^*_1 - \lambda'_2 \bar{e}^*_3)$;

• $\pi(\alpha_4(\lambda_1, \lambda_2)) = \lambda_1 \bar{e}^*_2 - \lambda_2 \bar{e}^*_1$ and $\pi(\alpha_4(\lambda_1, \lambda_2) \wedge \pi(\alpha_4(\lambda_1, \lambda_2))) = \lambda_1 \lambda_2 \cdot (\bar{e}^*_1 + \bar{e}^*_2) \wedge \bar{e}^*_3$;

• $\pi(\beta_1) = -2\bar{e}^*_1$ and $\pi(\beta_1 \wedge \pi(\beta_1)) = 2 \cdot \bar{e}^*_2 \wedge \bar{e}^*_3$;

• $\pi(\beta_2) = \bar{e}^*_2 - 2\bar{e}^*_1$ and $\pi(\beta_2 \wedge \pi(\beta_2)) = (\bar{e}^*_1 - 2\bar{e}^*_3) \wedge \bar{e}^*_2$.

Lemma 5.17 *The trivector β_1 is $Sp(V, f)$-equivalent with $\alpha_1(1, 1)$ if $char(\mathbb{F}) = 2$, and $Sp(V, f)$-equivalent with $\alpha_3(4, 2)$ if $char(\mathbb{F}) \neq 2$.*

Proof. Suppose $char(\mathbb{F}) = 2$. If θ is the element of $Sp(V, f)$ that maps the hyperbolic basis $(\bar{e}^*_1, \bar{f}^*_1, \bar{e}^*_2, \bar{f}^*_2, \bar{e}^*_3, \bar{f}^*_3)$ of (V, f) to the hyperbolic basis $(\bar{e}^*_1 + \bar{e}^*_3, \bar{f}^*_1 + \bar{f}^*_2, \bar{e}^*_1 + \bar{e}^*_2, \bar{f}^*_1 + \bar{f}^*_3, \bar{e}^*_1 + \bar{e}^*_2 + \bar{e}^*_3, \bar{f}^*_1 + \bar{f}^*_2 + \bar{f}^*_3)$ of (V, f), then $\bigwedge^3(\theta)$ maps $\alpha_1(1, 1) = \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_3 + \bar{e}^*_2 \wedge \bar{e}^*_3 \wedge \bar{f}^*_1 + \bar{e}^*_3 \wedge \bar{e}^*_1 \wedge \bar{f}^*_2$ to the trivector $\beta_1 = \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_2 + \bar{e}^*_2 \wedge \bar{e}^*_3 \wedge \bar{f}^*_1 + \bar{e}^*_3 \wedge \bar{e}^*_1 \wedge \bar{f}^*_3$.

Suppose $char(\mathbb{F}) \neq 2$. If θ is the element of $Sp(V, f)$ that maps the hyperbolic basis $(\bar{e}^*_1, \bar{f}^*_1, \bar{e}^*_2, \bar{f}^*_2, \bar{e}^*_3, \bar{f}^*_3)$ of (V, f) to the hyperbolic basis $(\bar{e}^*_1 - 2\bar{e}^*_2 + 2\bar{e}^*_3, \bar{f}^*_1 + \frac{1}{2}\bar{f}^*_2 + \frac{1}{2}\bar{f}^*_3, \bar{e}^*_1 - 2\bar{e}^*_2, -\bar{f}^*_1 - \bar{f}^*_2 - \frac{1}{2}\bar{f}^*_3, \bar{e}^*_1 - 2\bar{e}^*_3, \bar{f}^*_1 + \frac{1}{2}\bar{f}^*_2)$ of (V, f), then $\bigwedge^3(\theta)$ maps $\beta_1 = -\bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_2 + \bar{e}^*_2 \wedge \bar{e}^*_3 \wedge \bar{f}^*_1 + \bar{e}^*_3 \wedge \bar{e}^*_1 \wedge \bar{f}^*_3$ to $\alpha_3(4, 2) = \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_3 + 4 \cdot \bar{e}^*_2 \wedge \bar{e}^*_3 \wedge (\bar{f}^*_1 + \bar{f}^*_2 + \bar{f}^*_3) + \bar{e}^*_3 \wedge \bar{e}^*_1 \wedge (\bar{f}^*_2 + 2\bar{f}^*_3)$. ∎

Lemma 5.18 *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. If $\lambda_1 + \lambda_2 \neq 0$, then the trivector $\alpha_4(\lambda_1, \lambda_2)$ is $Sp(V, f)$-equivalent with $\alpha_2(\lambda_1 + \lambda_2, \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2})$. If $\lambda_1 + \lambda_2 = 0$, then the trivector $\alpha_4(\lambda_1, \lambda_2)$ is $Sp(V, f)$-equivalent with $\alpha_3(\lambda_1, 0)$.*

Proof. Suppose $\lambda_1+\lambda_2 \neq 0$. If θ is the element of $Sp(V,f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V,f) to the hyperbolic basis $(\bar{e}_1^*+\bar{e}_2^*, \frac{\lambda_1}{\lambda_1+\lambda_2}\bar{f}_1^*+\frac{\lambda_2}{\lambda_1+\lambda_2}\bar{f}_2^*, -\frac{\lambda_2}{\lambda_1+\lambda_2}\bar{e}_1^*+\frac{\lambda_1}{\lambda_1+\lambda_2}\bar{e}_2^*, -\bar{f}_1^*+\bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V,f), then $\bigwedge^3(\theta)$ maps $\alpha_2(\lambda_1+\lambda_2, \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + (\lambda_1+\lambda_2)\cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^*+\bar{f}_3^*) + \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ to $\alpha_4(\lambda_1,\lambda_2) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^*+\bar{f}_3^*) + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge (\bar{f}_2^*+\bar{f}_3^*)$.

Suppose $\lambda_2 = -\lambda_1$. If θ is the element of $Sp(V,f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V,f) to the hyperbolic basis $(\lambda_1\bar{e}_3^*, \bar{f}_1^*-\bar{f}_2^*+\frac{1}{\lambda_1}\bar{f}_3^*, \bar{e}_2^*+\lambda_1\bar{e}_3^*, \bar{f}_2^*, -\bar{e}_1^* +\lambda_1\bar{e}_3^*, -\bar{f}_1^*)$ of (V,f), then $\bigwedge^3(\theta)$ maps $\alpha_3(\lambda_1,0) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^*+\bar{f}_2^*+\bar{f}_3^*) - \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ to the trivector $\alpha_4(\lambda_1,-\lambda_1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^*+\bar{f}_3^*) - \lambda_1 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge (\bar{f}_2^*+\bar{f}_3^*)$. ∎

Lemma 5.19 *Let $\lambda_1,\lambda_2 \in \mathbb{F}^*$. If $char(\mathbb{F})=2$, then β_2 is not $Sp(V,f)$-equivalent with β_1, $\alpha_1(\lambda_1,\lambda_2)$, nor with $\alpha_2(\lambda_1,\lambda_2)$.*

Proof. If χ_1 and χ_2 are two trivectors of V which are $Sp(V,f)$-equivalent, then: (i) $\pi(\chi_1)$ and $\pi(\chi_2)$ are $Sp(V,f)$-equivalent; (ii) $\pi(\chi_1)\wedge\pi(\chi_1\wedge\pi(\chi_1))$ and $\pi(\chi_2)\wedge\pi(\chi_2\wedge\pi(\chi_2))$ are $Sp(V,f)$-equivalent.

Suppose $char(\mathbb{F})=2$. The facts that $\pi(\beta_2)\neq 0$, $\pi(\beta_1)=\pi(\alpha_1(\lambda_1,\lambda_2))=0$ imply that β_2 is not $Sp(V,f)$-equivalent with β_1 nor with $\alpha_1(\lambda_1,\lambda_2)$. Since $\pi(\beta_2)\wedge\pi(\beta_2\wedge\pi(\beta_2))=0$ and $\pi(\alpha_2(\lambda_1,\lambda_2))\wedge\pi(\alpha_2(\lambda_1,\lambda_2)\wedge\pi(\alpha_2(\lambda_1,\lambda_2)))\neq 0$, the trivector β_2 is also not $Sp(V,f)$-equivalent with $\alpha_2(\lambda_1,\lambda_2)$. ∎

Lemma 5.20 *If $char(\mathbb{F})\neq 2$, then the trivector β_2 is $Sp(V,f)$-equivalent with $\alpha_2(4,-\frac{1}{4})$. If $char(\mathbb{F})=2$ and $|\mathbb{F}|>2$, then the trivector β_2 is $Sp(V,f)$-equivalent with every trivector of the form $\alpha_3(\frac{1}{\eta^2+\eta},0)$, where $\eta\in\mathbb{F}\setminus\{0,1\}$. If $|\mathbb{F}|=2$ then β_2 is not $Sp(V,f)$-equivalent with a trivector of the form $\alpha_3(\lambda_1,\lambda_2)$ where $\lambda_1\in\mathbb{F}^*$ and $\lambda_2\in\mathbb{F}\setminus\{1\}$.*

Proof. Suppose $char(\mathbb{F})\neq 2$. If θ is the element of $Sp(V,f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V,f) to the hyperbolic basis $(2\bar{e}_1^*+2\bar{e}_2^*-4\bar{e}_3^*, \frac{1}{2}\bar{f}_1^*+\bar{f}_2^*+\frac{1}{2}\bar{f}_3^*, -\frac{1}{2}\bar{e}_1^*+\frac{1}{4}\bar{e}_2^*, -2\bar{f}_1^*-\bar{f}_3^*, \bar{e}_1^*+\frac{1}{2}\bar{e}_2^*-2\bar{e}_3^*, -\bar{f}_1^*-2\bar{f}_2^*-\frac{3}{2}\bar{f}_3^*)$ of (V,f), then $\bigwedge^3(\theta)$ maps $\alpha_2(4,-\frac{1}{4}) = \bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_3^* + 4\cdot\bar{e}_2^*\wedge\bar{e}_3^*\wedge(\bar{f}_1^*+\bar{f}_3^*) - \frac{1}{4}\cdot\bar{e}_3^*\wedge\bar{e}_1^*\wedge\bar{f}_2^*$ to the trivector $\beta_2 = -\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^* + \bar{e}_2^*\wedge\bar{e}_3^*\wedge(\bar{f}_1^*+\bar{f}_3^*) + \bar{e}_3^*\wedge\bar{e}_1^*\wedge\bar{f}_3^*$.

Suppose $char(\mathbb{F})=2$ and $\lambda\in\mathbb{F}^*$. If the trivectors β_2 and $\alpha_3(\lambda,0)$ are $Sp(V,f)$-equivalent, then for some hyperbolic bases $(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\bar{e}_3,\bar{f}_3)$ and $(\bar{e}_1',\bar{f}_1',\bar{e}_2',\bar{f}_2',\bar{e}_3',\bar{f}_3')$ of (V,f) we have $\beta_2' := \bar{e}_1'\wedge\bar{e}_2'\wedge\bar{f}_2' + \bar{e}_2'\wedge\bar{e}_3'\wedge(\bar{f}_1'+\bar{f}_3') + \bar{e}_3'\wedge\bar{e}_1'\wedge\bar{f}_3' = \alpha_3'(\lambda,0) := \bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_3 + \lambda\cdot\bar{e}_2\wedge\bar{e}_3\wedge(\bar{f}_1+\bar{f}_2+\bar{f}_3) + \bar{e}_3\wedge\bar{e}_1\wedge\bar{f}_2$. Let A, A' and M be the (3×3)-matrices as occurring in Lemma 5.3. Then $A = \begin{bmatrix} \lambda & \lambda & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Since $\pi(\beta_2') = \bar{e}_2' = \lambda\bar{e}_2+\lambda\bar{e}_3 = \pi(\alpha_3'(\lambda,0))$ and $\pi(\beta_2'\wedge\pi(\beta_2')) = \bar{e}_1'\wedge\bar{e}_2' = \lambda\cdot\bar{e}_1\wedge(\bar{e}_2+\bar{e}_3) = \pi(\alpha_3'(\lambda,0)\wedge\pi(\alpha_3'(\lambda,0)))$, we have $M = \begin{bmatrix} 1 & t & t \\ 0 & \lambda & \lambda \\ s_1 & s_2 & s_3 \end{bmatrix}$ for some $t,s_1,s_2,s_3\in\mathbb{F}$.

By Lemma 5.3 we have that $det(M)A' = MAM^T$. From $det(M)A'_{11} = (MAM^T)_{11}$ we obtain $s_3 = s_2 + 1$, from $det(M)A'_{13} = (MAM^T)_{13}$ we obtain $t = \lambda s_1$ and from $det(M)A'_{33} = (MAM^T)_{33}$ we obtain $\lambda s_1^2 + \lambda s_1 + 1 = 0$. These equations cannot have a solution if $|\mathbb{F}| = 2$. So, if $|\mathbb{F}| = 2$, then β_2 is not $Sp(V, f)$-equivalent with a trivector of the form $\alpha_3(\lambda, 0)$ with $\lambda \in \mathbb{F}^*$. On the other hand, if $|\mathbb{F}| > 2$ then $det(M)A' = MAM^T$ if we choose $s_1 \in \mathbb{F} \setminus \{0, 1\}$, $s_2 = 1$, $s_3 = 0$, $t = \frac{1}{s_1+1}$ and $\lambda = \frac{1}{s_1^2+s_1}$. So, by Lemma 5.4, we know that if $|\mathbb{F}| > 2$, then β_2 is $Sp(V, f)$-equivalent with every trivector of the form $\alpha_3(\frac{1}{\eta^2+\eta}, 0)$, where η is an element of $\mathbb{F} \setminus \{0, 1\}$.

Suppose $|\mathbb{F}| = 2$. Suppose β_2 is $Sp(V, f)$-equivalent with the trivector $\alpha_3(\lambda_1, \lambda_2)$ where λ_1 and λ_2 are elements of $\mathbb{F}$ such that $\lambda_1 \neq 0$ and $\lambda_2 \neq 1$. By the previous paragraph, we then know that $\lambda_2 \neq 0$. But then $\pi(\alpha_3(\lambda_1, \lambda_2)) \wedge \pi(\alpha_3(\lambda_1, \lambda_2) \wedge \pi(\alpha_3(\lambda_1, \lambda_2))) \neq 0$, which is impossible since $\pi(\beta_2) \wedge \pi(\beta_2 \wedge \pi(\beta_2)) = 0$. ∎

Lemma 5.21 *Let $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$ and $\lambda''_2 \in \mathbb{F} \setminus \{1\}$. Then the trivector $\alpha_1(\lambda_1, \lambda_2)$ is neither $Sp(V, f)$-equivalent with $\alpha_2(\lambda'_1, \lambda'_2)$ nor with $\alpha_3(\lambda'_1, \lambda''_2)$.*

Proof. If χ_1 and χ_2 are two $Sp(V, f)$-equivalent trivectors, then also $\pi(\chi_1)$ and $\pi(\chi_2)$ are $Sp(V, f)$-equivalent. The lemma then follows from the facts that $\pi(\alpha_1(\lambda_1, \lambda_2)) = 0$, $\pi(\alpha_2(\lambda'_1, \lambda'_2)) \neq 0$ and $\pi(\alpha_3(\lambda'_1, \lambda''_2)) \neq 0$. ∎

The following lemma is a consequence of Corollary 5.5.

Lemma 5.22 *Let $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{F}^*$. Then the trivectors $\alpha_1(\lambda_1, \lambda_2)$ and $\alpha_1(\lambda'_1, \lambda'_2)$ are $Sp(V, f)$-equivalent if and only if the matrices $\begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_1\lambda_2} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\lambda'_1} & 0 & 0 \\ 0 & \frac{1}{\lambda'_2} & 0 \\ 0 & 0 & \frac{1}{\lambda'_1\lambda'_2} \end{bmatrix}$ are congruent.*

Lemma 5.23 *Let $\mu_1, \mu_2 \in \mathbb{F}$ such that $\mu_1 \neq 0$ and $\mu_2 \neq 1$. Then the trivector $\alpha_3(\mu_1, \mu_2)$ is $Sp(V, f)$-equivalent with a trivector of the form $\alpha_2(\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbb{F}^*$ if and only if $\mu_2 \neq 0$ and $(\mu_1, \mu_2) \neq (4, 2)$.*

Proof. If $\mu_2 = 0$, then $\pi(\alpha_3(\mu_1, 0)) \wedge \pi(\alpha_3(\mu_1, 0) \wedge \pi(\alpha_3(\mu_1, 0))) = 0$ and $\pi(\alpha_2(\lambda_1, \lambda_2)) \wedge \pi(\alpha_2(\lambda_1, \lambda_2) \wedge \pi(\alpha_2(\lambda_1, \lambda_2))) \neq 0$ for all $\lambda_1, \lambda_2 \in \mathbb{F}^*$. So, the trivector $\alpha_3(\mu_1, 0)$ is not $Sp(V, f)$-equivalent with a trivector of the form $\alpha_2(\lambda_1, \lambda_2)$ where $\lambda_1, \lambda_2 \in \mathbb{F}^*$. In the sequel, we will suppose that $\mu_2 \neq 0$.

Suppose that $\alpha_3(\mu_1, \mu_2)$ and $\alpha_2(\lambda_1, \lambda_2)$ are $Sp(V, f)$-equivalent for some $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then for some hyperbolic bases $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ and $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ of (V, f), we have $\alpha'_2(\lambda_1, \lambda_2) := \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3 + \lambda_1 \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge (\bar{f}'_1 + \bar{f}'_3) + \lambda_2 \cdot \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_2 = \alpha'_3(\mu_1, \mu_2) := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \mu_1 \cdot \bar{e}_2 \wedge \bar{e}_3 \wedge (\bar{f}_1 + \bar{f}_2 + \bar{f}_3) + \bar{e}_3 \wedge \bar{e}_1 \wedge ((\mu_2 - 1)\bar{f}_2 + \mu_2 \bar{f}_3)$. Let A, A' and M be the (3×3)-matrices as occurring in Lemma 5.3. Then $A = \begin{bmatrix} \mu_1 & \mu_1 & \mu_1 \\ 0 & \mu_2 - 1 & \mu_2 \\ 0 & 0 & 1 \end{bmatrix}$ and $A' =$

$\begin{bmatrix} \lambda_1 & 0 & \lambda_1 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We have $\pi(\alpha_2'(\lambda_1,\lambda_2)) = \lambda_1 \bar{e}_2' = \mu_1\bar{e}_2 - \mu_1\bar{e}_3 - \mu_2\bar{e}_1 = \pi(\alpha_3'(\mu_1,\mu_2))$ and $\pi(\alpha_2'(\lambda_1,\lambda_2)\wedge\pi(\alpha_2'(\lambda_1,\lambda_2))) = \lambda_1\lambda_2\cdot\bar{e}_1'\wedge\bar{e}_3' = \frac{\mu_1}{\mu_2}\cdot(\bar{e}_1-\mu_2\bar{e}_2)\wedge(\bar{e}_1-\mu_2\bar{e}_3) = \pi(\alpha_3'(\mu_1,\mu_2)\wedge \pi(\alpha_3'(\mu_1,\mu_2)))$. From those two equations we have that $M = \begin{bmatrix} s_1+t_1 & -s_1\mu_2 & -t_1\mu_2 \\ -\frac{\mu_2}{\lambda_1} & \frac{\mu_1}{\lambda_1} & -\frac{\mu_1}{\lambda_1} \\ s_2+t_2 & -s_2\mu_2 & -t_2\mu_2 \end{bmatrix}$ for some $t_1,s_1,t_2,s_2\in\mathbb{F}$ such that $s_1t_2-s_2t_1\neq 0$. We have $det(M) = \frac{\mu_2^3}{\lambda_1}(s_1t_2-s_2t_1)$. By Lemma 5.3, we have $det(M)A' = MAM^T$. So, $det(M) = \frac{det(A)}{det(A')} = \frac{\mu_1(\mu_2-1)}{\lambda_1\lambda_2}$. From $det(M)A'_{22} = (MAM^T)_{22}$, we have $\frac{\mu_1(\mu_2-1)}{\lambda_1} = \frac{\mu_2^2\mu_1}{\lambda_1^2}$. So, $\lambda_1 = \frac{\mu_2^2}{\mu_2-1}$.

If $\mu_1\neq\mu_2^2$, then $det(M)A' = MAM^T$ if we put $s_1 = \frac{\mu_2^3-\mu_1\mu_2+\mu_1}{\mu_2^2(\mu_2-1)(\mu_1-\mu_2^2)}$, $t_1 = \frac{1}{\mu_2^2}$, $s_2 = \frac{1}{\mu_2(\mu_1-\mu_2^2)}$, $t_2 = 0$ and $\lambda_2 = \mu_1(\mu_2-1)(\mu_2^2-\mu_1)$. So, in this case $\alpha_3(\mu_1,\mu_2)$ is $Sp(V,f)$-equivalent with $\alpha_2(\frac{\mu_2^2}{\mu_2-1},\mu_1(\mu_2-1)(\mu_2^2-\mu_1))$ by Lemma 5.4.

If $\mu_1=\mu_2^2$ and $\mu_2\neq 2$, then $det(M)A' = MAM^T$ if we put $s_1 = -\frac{\mu_2^2-\mu_2+1}{\mu_2}$, $t_1=\mu_2$, $s_2 = -\frac{2\mu_2-1}{\mu_2^2}$, $t_2=1$ and $\lambda_2 = -\frac{1}{\mu_2-2}$. So, in this case $\alpha_3(\mu_1,\mu_2)$ is $Sp(V,f)$-equivalent with $\alpha_2(\frac{\mu_2^2}{\mu_2-1},-\frac{1}{\mu_2-2})$ by Lemma 5.4.

Finally, suppose that $char(\mathbb{F})\neq 2$ and $(\mu_1,\mu_2)=(4,2)$. Then $(MAM^T)_{33}=0$ and $det(M)\cdot A'_{33} = \det(M)\neq 0$. So, by Lemma 5.3, the trivector $\alpha_3(4,2)$ cannot be $Sp(V,f)$-equivalent with a trivector of the form $\alpha_2(\lambda_1,\lambda_2)$ where $\lambda_1,\lambda_2\in\mathbb{F}^*$. ■

Lemma 5.24 *Let $\lambda,\mu\in\mathbb{F}^*$. If $char(\mathbb{F})\neq 2$, then the trivectors $\alpha_3(4,2)$ and $\alpha_3(\lambda,0)$ are not $Sp(V,f)$-equivalent. If $char(\mathbb{F})=2$, then the trivectors $\alpha_3(\lambda,0)$ and $\alpha_3(\mu,0)$ are $Sp(V,f)$-equivalent if and only if $\frac{\lambda+\mu}{\lambda\mu}$ is of the form s^2+s for some $s\in\mathbb{F}$. If $char(\mathbb{F})\neq 2$, then the trivectors $\alpha_3(\lambda,0)$ and $\alpha_3(\mu,0)$ are always $Sp(V,f)$-equivalent.*

Proof. If $char(\mathbb{F})\neq 2$, then since $\pi(\alpha_3(\lambda,0))\wedge\pi(\alpha_3(\lambda,0)\wedge\pi(\alpha_3(\lambda,0)))=0$ and $\pi(\alpha_3(4,2))\wedge\pi(\alpha_3(4,2)\wedge\pi(\alpha_3(4,2)))\neq 0$, the trivectors $\alpha_3(4,2)$ and $\alpha_3(\lambda,0)$ cannot be $Sp(V,f)$-equivalent.

Suppose that $\alpha_3(\mu,0)$ and $\alpha_3(\lambda,0)$ are $Sp(V,f)$-equivalent. Then for some hyperbolic bases $(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\bar{e}_3,\bar{f}_3)$ and $(\bar{e}_1',\bar{f}_1',\bar{e}_2',\bar{f}_2',\bar{e}_3',\bar{f}_3')$ of (V,f), we have $\alpha_3'(\lambda,0) = \bar{e}_1'\wedge\bar{e}_2'\wedge\bar{f}_3'+\lambda\cdot\bar{e}_2'\wedge\bar{e}_3'\wedge(\bar{f}_1'+\bar{f}_2'+\bar{f}_3') - \bar{e}_3'\wedge\bar{e}_1'\wedge\bar{f}_2' = \alpha_3''(\mu,0) := \bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_3+\mu\cdot\bar{e}_2\wedge\bar{e}_3\wedge(\bar{f}_1+\bar{f}_2+\bar{f}_3)-\bar{e}_3\wedge\bar{e}_1\wedge\bar{f}_2$. Let A, A' and M be the (3×3)-matrices as occurring in Lemma 5.3. Then $A = \begin{bmatrix} \mu & \mu & \mu \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A' = \begin{bmatrix} \lambda & \lambda & \lambda \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We have that $\pi(\alpha_3'(\lambda,0)) = \lambda\bar{e}_2'-\lambda\bar{e}_3' = \mu\bar{e}_2-\mu\bar{e}_3 = \pi(\alpha_3''(\mu,0))$ and $\pi(\alpha_3'(\lambda,0)\wedge\pi(\alpha_3'(\lambda,0))) = \lambda\cdot\bar{e}_1'\wedge(\bar{e}_2'-\bar{e}_3') = \mu\cdot\bar{e}_1\wedge(\bar{e}_2-\bar{e}_3) = \pi(\alpha_3''(\mu,0)\wedge\pi(\alpha_3''(\mu,0)))$. From those two equations we have that $M = \begin{bmatrix} 1 & k & -k \\ s_1 & s_2+\frac{\mu}{\lambda} & s_3-\frac{\mu}{\lambda} \\ s_1 & s_2 & s_3 \end{bmatrix}$ for some $k,s_1,s_2,s_3\in\mathbb{F}$. By Lemma 5.3 we have that $det(M)A' = MAM^T$, or equivalently, $s_2+s_3=1$, $k=s_1\mu$ and $2s_3 = \frac{\mu}{\lambda}+1-s_1^2\mu-s_1\mu$.

If $char(\mathbb{F}) = 2$, then the latter condition becomes $\frac{\lambda+\mu}{\lambda\mu} = s_1^2 + s_1$. So, if $char(\mathbb{F}) = 2$ then the fact that $\alpha_3(\lambda, 0)$ and $\alpha_3(\mu, 0)$ are $Sp(V, f)$-equivalent implies that $\frac{\lambda+\mu}{\lambda\mu} = s^2 + s$ for some $s \in \mathbb{F}$.

Conversely, if $char(\mathbb{F}) = 2$ and there exists an $s \in \mathbb{F}$ such that $\frac{\lambda+\mu}{\lambda\mu} = s^2 + s$, then $det(M)A' = MAM^T$ if we put $s_1 := s$, $s_3 := 1 + s_2$ and $k := \mu s$ for some arbitrary $s_2 \in \mathbb{F}$. So, if such an s exists, then the trivectors $\alpha_3(\lambda, 0)$ and $\alpha_3(\mu, 0)$ are $Sp(V, f)$-equivalent by Lemma 5.4.

If $char(\mathbb{F}) \neq 2$, then $det(M)A' = MAM^T$ if we put $s_3 := \frac{1}{2}(\frac{\mu}{\lambda} + 1 - s_1^2\mu - s_1\mu)$, $s_2 := 1 - s_3$ and $k := s_1\mu$ for some arbitrary $s_1 \in \mathbb{F}$. So, if $char(\mathbb{F}) \neq 2$, then the trivectors $\alpha_3(\lambda, 0)$ and $\alpha_3(\mu, 0)$ are always $Sp(V, f)$-equivalent by Lemma 5.4. ∎

Lemma 5.25 *Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{F}^*$. Then the trivectors $\alpha_2(\lambda_1, \lambda_2)$ and $\alpha_2(\mu_1, \mu_2)$ are $Sp(V, f)$-equivalent if and only if $\lambda_1 = \mu_1$ and there exist $s, t \in \mathbb{F}$ such that $t^2 + st\lambda_1 + s^2\lambda_1 = \frac{\mu_2}{\lambda_2}$.*

Proof. Suppose that $\alpha_2(\mu_1, \mu_2)$ and $\alpha_2(\lambda_1, \lambda_2)$ are $Sp(V, f)$-equivalent. Then for some hyperbolic bases $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ and $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ of (V, f), we have $\alpha'_2(\lambda_1, \lambda_2) := \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_3 + \lambda_1 \cdot \bar{e}'_2 \wedge \bar{e}'_3 \wedge (\bar{f}'_1 + \bar{f}'_3) + \lambda_2 \cdot \bar{e}'_3 \wedge \bar{e}'_1 \wedge \bar{f}'_2 = \alpha''_2(\mu_1, \mu_2) := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \mu_1 \cdot \bar{e}_2 \wedge \bar{e}_3 \wedge (\bar{f}_1 + \bar{f}_3) + \mu_2 \cdot \bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_2$. Let A, A' and M be the (3×3)-matrices as occurring in Lemma 5.3. Then $A = \begin{bmatrix} \mu_1 & 0 & \mu_1 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A' = \begin{bmatrix} \lambda_1 & 0 & \lambda_1 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We have that $\pi(\alpha'_2(\lambda_1, \lambda_2)) = \lambda_1\bar{e}'_2 = \mu_1\bar{e}_2 = \pi(\alpha''_2(\mu_1, \mu_2))$ and $\pi(\alpha'_2(\lambda_1, \lambda_2) \wedge \pi(\alpha'_2(\lambda_1, \lambda_2))) = \lambda_1\lambda_2 \cdot \bar{e}'_1 \wedge \bar{e}'_3 = \mu_1\mu_2 \cdot \bar{e}_1 \wedge \bar{e}_3 = \pi(\alpha''_2(\mu_1, \mu_2) \wedge \pi(\alpha''_2(\mu_1, \mu_2)))$. From those two equations, we obtain that $M = \begin{bmatrix} s_1 & 0 & t_1 \\ 0 & \frac{\mu_1}{\lambda_1} & 0 \\ s_2 & 0 & t_2 \end{bmatrix}$ for some $s_1, s_2, t_1, t_2 \in \mathbb{F}$. By Lemma 5.3, we have $det(M)A' = MAM^T$. We also have $(s_1t_2 - s_2t_1) \cdot \frac{\mu_1}{\lambda_1} = \det(M) = \frac{\det(A)}{\det(A')} = \frac{\mu_1\mu_2}{\lambda_1\lambda_2}$. Combining this with $det(M)A'_{22} = (MAM^T)_{22}$ we obtain $\lambda_1 = \mu_1$ and $\det(M) = s_1t_2 - s_2t_1 = \frac{\mu_2}{\lambda_2}$. From $t_1 \cdot det(M)A'_{33} - t_2 \cdot det(M)A'_{31} = t_1 \cdot (MAM^T)_{33} - t_2 \cdot (MAM^T)_{31}$, we obtain that $t_1 = -s_2\lambda_1$. From $t_1 \cdot det(M)A'_{13} - t_2 \cdot det(M)A'_{11} = t_1 \cdot (MAM^T)_{13} - t_2 \cdot (MAM^T)_{11}$, we obtain that $s_1 = t_2 + s_2\lambda_1$. The condition $s_1t_2 - s_2t_1 = \frac{\mu_2}{\lambda_2}$ $(= det(M))$, then becomes $t_2^2 + s_2t_2\lambda_1 + s_2^2\lambda_1 = \frac{\mu_2}{\lambda_2}$.

Conversely, if $\lambda_1 = \mu_1$ and $s_1, t_1, s_2, t_2 \in \mathbb{F}$ are such that $t_2^2 + s_2t_2\lambda_1 + s_2^2\lambda_1 = \frac{\mu_2}{\lambda_2}$, $t_1 = -s_2\lambda_1$ and $s_1 = t_2 + s_2\lambda_1$, then $\det(M)A' = MAM^T$. So, by Lemma 5.4, the two trivectors $\alpha_2(\lambda_1, \lambda_2)$ and $\alpha_2(\mu_1, \mu_2)$ are $Sp(V, f)$-equivalent if and only if $\lambda_1 = \mu_1$ and there exist $s, t \in \mathbb{F}$ such that $t^2 + st\lambda_1 + s^2\lambda_1 = \frac{\mu_2}{\lambda_2}$. ∎

We now observe the following:

- For all $\lambda_1, \lambda_2 \in \mathbb{F}^*$, we have $\alpha_1(\lambda_1, \lambda_2) = \gamma_3(\lambda_1, \lambda_2)$.
- For all $\lambda_1, \lambda_2 \in \mathbb{F}^*$, we have $\alpha_2(\lambda_1, \lambda_2) = \gamma_4(\lambda_1, \lambda_2)$.
- For all $\lambda \in \mathbb{F}^*$, we have $\alpha_3(\lambda, 0) = \gamma_5(\lambda)$.
- If $char(\mathbb{F}) \neq 2$, then $\beta_1 = \gamma_6$.

- If $|\mathbb{F}| = 2$, then $\beta_2 = \gamma_7$.

Theorems 1.2 and 1.3 are a consequence of Lemmas 5.16-5.25 and the above observations.

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